Particle Currents

This document is a summary of the current state of work on quantum mechanics in phase space. It goes over the preliminaries for investigating whether Wigner functions can be interpreted as time components of a particle current.

Basic ideas

Quantum mechanics can be formulated in phase space, using Wigner functions rather than wave functions as the mathematical objects containing all the information about the state of a particle. A Wigner function $f(x,p)$ is such that it behaves exactly like a probability density in every respect. It is normalized:

$$\int dx\, dp\, f(x,p) = 1 \tag{1}$$

and the expectation value of *any* physical observable $Q(x,p)$ is given as

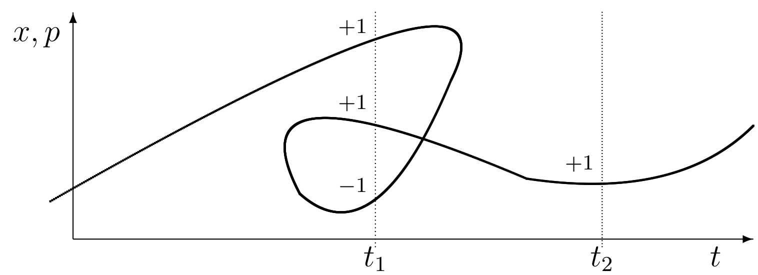
$$\int dx\, dp\, f(x,p)\, Q(x,p) = \langle Q \rangle \tag{2}$$

However, $f(x,p)$ is not a proper probability distribution because it can be negative: $f(x,p) < 0$ is true in parts of phase space for many particle states.

So $f(x,p)$ is like a probability distribution in that it is normalized, but the non-negativity constraint on probability distributions is relaxed. But there is another complication. Not every normalized distribution is a possible Wigner function, since many possible distributions violate the uncertainty principle. For example, consider a particle completely localized in phase space, with $x = x_0$ and $p = p_0$. In that case, $\Delta x = \Delta p = 0$, and therefore the distribution $f(x,p) = \delta(x - x_0)\delta(p - p_0)$ violates the uncertainty principle. Only a small subset of all normalized distributions can represent quantum states and are legal Wigner functions.

Our research is aimed at finding a natural physical interpretation for a Wigner function in terms of a particle moving through phase space. The particle will be doing a random walk, or, more properly speaking, a generalization of a random walk applied to a continuous underlying space such as phase space. So each particle trajectory in phase space will have a probability associated with it. As the particle moves, it can be described in terms of a probability distribution in phase space. But as the previous discussion makes clear, this cannot be equated to a Wigner function. We need to something tricky.

The basic idea is that we will allow the particle to move backward as well as forward in time, relaxing the constraint on particle trajectories that they are single-valued functions of time t. Then, instead of a particle density which is constrained to be non-negative, we have to speak of a particle *current*, which can be negative in those regions of phase space where the particle has a tendency to move backwards in time. If all particle trajectories start at $t = -\infty$ and extend to $t = +\infty$, the total particle current at any given time is always 1, as seen in the picture:



At any given time, say t_1 or t_2, the current contributions of the forward-moving particles $(+1)$ and the backward-moving particles (-1) always add up to $+1$. Therefore the time-component of the particle current will be normalized to 1, like a Wigner function.

Travel back in time will also have an effect of spreading out the particle, so that at any given time, the particle cannot be definitely localized. This is because backward-looping trajectories have a non-zero probability, and so we cannot definitely know that a particle is located at (x_0, p_0) at any given time t. It can also be moving forward at (x_1, p_1) and backward at (x_2, p_2), and *all* such possibilities need to be taken into account. This *might* lead to the exclusion of currents that violate the uncertainty principle, which is what we want for Wigner functions.

In this picture, the time component of the particle current can be negative in certain regions of phase space. With Wigner functions, this is a sign of specially quantum effects, due to wave interference in wave mechanics or the Hilbert space/wavefunction formulation of quantum mechanics. *If* our picture works out, these quantum effects will be due to time travel, not some sort of wave-particle duality.

Currents

We now need some mathematical apparatus to deal with currents for particles that can move backwards in time. Since particle trajectories are no longer single-valued functions of t, we describe them as parametric curves, with the particle location $x_\pi(\tau)$, momentum $p_\pi(\tau)$, and time $t_\pi(\tau)$—all functions of a parameter τ. (Think of τ as a kind of internal time for the particle, which is distinct from external time t. The particle label is "π" just to prevent confusion with momentum p.)

For simplicity, let us start with the deterministic case: x_π, p_π, and t_π are functions of τ as described. At any given τ, the probability density for the particle is

$$\rho = \delta[x - x_\pi(\tau)]\delta[p - p_\pi(\tau)]\delta[t - t_\pi(\tau)] \tag{3}$$

In other words, all delta-functions because we know exactly where the particle is. Now, for a single particle on a definite trajectory the current is $\vec{\jmath} = \rho\,\vec{v}$, the density times the velocity. In our case, "time" is the parameter τ, while t is like another spatial coordinate. For the time component of the current, therefore, we need the time component of the velocity. These components are

$$v_x = \frac{dx_\pi}{d\tau} \qquad v_\pi = \frac{dp_\pi}{d\tau} \qquad v_t = \frac{dt_\pi}{d\tau} \tag{4}$$

Picking out v_t, we can write the contribution of the particle to the total time-current j_t at a given τ:

$$j_t(\tau) = \rho v_t = \delta[x - x_\pi(\tau)]\delta[p - p_\pi(\tau)]\delta[t - t_\pi(\tau)]\,\frac{dt_\pi}{d\tau} \tag{5}$$

The parameter τ is not physically observable; we are interested in finding the *total* contribution from the particles' trajectory. So we have to integrate over all τ values.

$$J_t = \int_{-\infty}^{\infty} d\tau\, j_t(\tau) \tag{6}$$

For example, consider a case where the particle does not loop back in time, and $t_\pi = \tau$. Performing the integral, we find $J_t = \delta[x - x_\pi(\tau)]\delta[p - p_\pi(\tau)]$, which is the ordinary particle density, as it should be for that case.

Now, let us generalize to a particle doing a random walk through phase space. Say that at a given value of τ, the particle is located at (x, p, t). Then, it will have a probability of moving to location (x', p', t') a distance $d\tau$ along the trajectory. In other words, there is a transition probability density $P_T(x, p, t \,|\, x', p', t')$ describing the random walk. Note that this uses the notation of conditional probability; it expresses the probability of the particle being at (x, p, t) at point $\tau + d\tau$ along its trajectory, *given that* it was at (x', p', t') at point τ. P_T does not depend on τ, since that is an arbitrary parameter.

We also need to introduce the probability distribution on phase space for a particular parameter τ. Call this $P_\pi(x, p, t; \tau)$: the probability that the particle has a particular location, momentum, and *time* value at a given τ.

Now, let's relate P_T and P_π. We look at infinitesimal differences in the parameter τ, or $d\tau$. In this case

$$P_\pi(x, p, t; \tau + d\tau) = P_\pi(x, p, t; \tau) + \frac{dP_\pi}{d\tau}\, d\tau \tag{7}$$

$$P_\pi(x, p, t; \tau + d\tau) = \int dx'\, dp'\, dt'\, P_T(x, p, t \,|\, x', p', t') P_\pi(x', p', t'; \tau) \tag{8}$$

Equation (7) comes from basic calculus. (8) is due to conditional probability: it adds up the probabilities of ending up at (x, p, t) for all possible starting points (x', p', t'). To make (8) look more like (7), we can say

$$P_T(x, p, t \,|\, x', p', t') = \delta(x - x')\delta(p - p')\delta(t - t') + T(x, p, t \,|\, x', p', t')\, d\tau \tag{9}$$

This just expresses the condition that as $d\tau \to 0$, there should be no change in the particle's position, which is enforced by the leading δ-functions. All the interesting information about the transition probabilities, in other words, are contained in the transition matrix T.

Peforming the integral, (8) becomes

$$\begin{aligned} P_\pi(x, p, t; \tau + d\tau) \;&=\; P_\pi(x, p, t; \tau) + \\ &\quad d\tau \int dx'\, dp'\, dt'\, T(x, p, t \,|\, x', p', t') P_\pi(x', p', t'; \tau) \end{aligned} \tag{10}$$

Note that (10) involves the *matrix* T, though T has continuous indices. Remember that for a discrete matrix M_{ij} and vector A_j, matrix multiplication looks like

$$B_i = \sum_j M_{ij} A_j \qquad \text{or} \qquad B = \hat{M} A \tag{11}$$

where A and B are vectors. Something similar to our case would be if

$$B_i = \sum_j (\delta_{ij} + \epsilon T_{ij}) A_j \qquad \text{or} \qquad B = (\hat{I} + \epsilon \hat{T}) A \tag{12}$$

where ϵ is a number. Notice how (9) is just a continuously indexed version of the same sort of equation, with the role of ϵ played by $d\tau$. In fact, we can rewrite (9) as

$$\hat{P}_T = \hat{I} + \hat{T}\, d\tau \tag{13}$$

In the same notation, equation (10) becomes

$$P_\pi(\tau + d\tau) = P_\pi(\tau) + d\tau\, \hat{T} P_\pi(\tau) = (\hat{I} + d\tau\, \hat{T}) P_\pi(\tau) \tag{14}$$

The transition operator $\hat{T}$ has "matrix elements" $T(x, p, t \,|\, x', p', t')$.

As an example, again consider the deterministic case, with $P_\pi = \rho$, expressed in (3). Combining (7) and (10), we have

$$\frac{dP_\pi}{d\tau} = \int dx'\, dp'\, dt'\, T(x, p, t \,|\, x', p', t') P_\pi(x', p', t'; \tau) \tag{15}$$

In our deterministic test case, doing the derivative on the left side and the integration on the right,

$$-\frac{dx_\pi}{d\tau}\, \delta'(x - x_\pi)\delta(p - p_\pi)\delta(t - t_\pi) - \frac{dp_\pi}{d\tau}\, \delta(x - x_\pi)\delta'(p - p_\pi)\delta(t - t_\pi) -$$
$$\frac{dt_\pi}{d\tau}\, \delta(x - x_\pi)\delta(p - p_\pi)\delta'(t - t_\pi) = T(x, p, t \,|\, x_\pi, p_\pi, t_\pi) \tag{16}$$

In other words, an equation for the transition matrix involving derivatives of the delta function, δ'.

Knowing how change from τ to $\tau + d\tau$ happens, we can also obtain how P_π changes over finite intervals τ to $\tau + \Delta\tau$. Consider a small interval τ to $\tau + \epsilon$. If ϵ is small, then, to first order in ϵ,

$$P_\pi(\tau + \epsilon) = (\hat{I} + \epsilon\, \hat{T}) P_\pi(\tau) + O(\epsilon^2) \tag{17}$$

But we can also change from τ to $\tau + \epsilon/2$ first and then to $\tau + \epsilon$. In this two-stage process,

$$P_\pi(\tau + \epsilon) = \left(\hat{I} + \frac{\epsilon}{2}\, \hat{T}\right)^2 P_\pi(\tau) + \ldots \tag{18}$$

If we generalize this,

$$P_\pi(\tau + \epsilon) = \left(\hat{I} + \frac{\epsilon}{N}\, \hat{T}\right)^N P_\pi(\tau) + \ldots \tag{19}$$

Now, as $N \to \infty$, $\frac{\epsilon}{N} \to 0$ and the first order approximation becomes exact. So

$$P_\pi(\tau + \epsilon) = \lim_{N\to\infty} \left(\hat{I} + \frac{\epsilon}{N}\hat{T}\right)^N P_\pi(\tau) = e^{\epsilon \hat{T}} P_\pi(\tau) \tag{20}$$

In other words, the operator $\exp(\Delta\tau\,\hat{T})$ acts such that for finite $\Delta\tau$,

$$P_\pi(\tau + \Delta\tau) = e^{\Delta\tau\,\hat{T}} P_\pi(\tau) \tag{21}$$

This is actually a compact expression of a path integral. Recall (8) and (10): written out in terms of integrals, these steps give

$$\begin{aligned} &P_\pi(x, p, t; \tau + \epsilon) = \\ &\int dx'\, dp'\, dt'\, \left[\delta(x - x')\delta(p - p')\delta(t - t') + \epsilon\, T(x, p, t \,|\, x', p', t')\right] P_\pi(x', p', t'; \tau) \\ &\qquad + O(\epsilon^2) \end{aligned} \tag{22}$$

And in terms of N successive steps of $\frac{\epsilon}{N}$ size each,

$$\begin{aligned} P_\pi(x^{(N+1)}, p^{(N+1)}, t^{(N+1)}; \tau + \epsilon) &= \int dx^{(N)} dp^{(N)} dt^{(N)} \ldots \int dx^{(1)} dp^{(1)} dt^{(1)} \\ &\Pi_{n=1}^{N} \left[\delta(x^{(n+1)} - x^{(n)})\delta(p^{(n+1)} - p^{(n)})\delta(t^{(n+1)} - t^{(n)}) + \right. \\ &\left. \frac{\epsilon}{N}\, T(x^{(n+1)}, p^{(n+1)}, t^{(n+1)} \,|\, x^{(n)}, p^{(n)}, t^{(n)})\right] P_\pi(x^{(1)}, p^{(1)}, t^{(1)}; \tau) \end{aligned} \tag{23}$$

This complicated expression, when $N \to \infty$, defines a "path integral" over all possible trajectories connecting (x, p, t) at $\tau + \epsilon$ and (x', p', t') at τ. Path integrals are *very* difficult to calculate directly.

We are not interested in finding the P_π distribution except as a tool to obtain the current—after all, τ is just a parameter, and P_π at a given τ is not a physically meaningful quantity.

So, let us say we knew $P_\pi(\tau)$, and we wanted to find the current component J_t as in (6). We then can also obtain the distribution $P_\pi(\tau + d\tau)$. If we add up all the current contributions due to transitions that take place from τ to $\tau + d\tau$, we can construct $j_t(\tau)$ as in (5).

If the particle was located at (x', p', t') at τ, and it moves to (x, p, t) at $\tau + d\tau$, the contribution to current due to this transition would be

$$P_\pi(x', p', t'; \tau) P_T(x, p, t \,|\, x', p', t') \frac{dt}{d\tau} \tag{24}$$

Here, $\frac{dt}{d\tau} = (t - t')/\epsilon$ for infinitesimal ϵ, so let us use ϵ for $d\tau$ for a while. We need to add contributions from all possible starting points (x', p', t'), so we integrate:

$$\begin{aligned} j_t(\tau) &= \int dx'\, dp'\, dt'\, P_\pi(x', p', t'; \tau)\, [\delta(x - x')\delta(p - p')\delta(t - t') + \\ &\quad \epsilon\, T(x, p, t \,|\, x', p', t')]\, \frac{t - t'}{\epsilon} \end{aligned} \tag{25}$$

The term with the δ functions gives zero when integrated, since then $t = t'$ and there is no change along the time axis. So we are left with

$$j_t(x, p, t; \tau) = \int dx'\, dp'\, dt'\, P_\pi(x', p', t'; \tau)\, T(x, p, t \,|\, x', p', t')\, (t - t') \tag{26}$$

For the total current, we integrate j_t over all τ:

$$J_t(x, p, t) = \int d\tau \int dx'\, dp'\, dt'\, P_\pi(x', p', t'; \tau)\, T(x, p, t \,|\, x', p', t')\, (t - t') \tag{27}$$

So, if we are to see if J_t has the properties of a Wigner function, we need some way to get a handle on it.

Research projects

There are a number of questions that need to be resolved, which might be the basis for individual student research projects.

The allowed probability distributions

We need to have an idea of P_π, which in (23) is a nasty path integral, but looks more tractable in the operator notation of (21).

We have one very important simplification available to us: we are interested in *infinite* intervals $\Delta\tau$. That is, we start at an arbitrary initial distribution at $\tau = -\infty$;

$$P_\pi(x, p, t; -\infty) = \delta(t + \infty) P_0(x, p) \tag{28}$$

We then need to bring this distribution to finite values of τ. Because of the spreading effects of the random walk over an infinite parameter interval, only a limited set of possible distributions $\{P_\pi\}$ will survive in this infinite interval limit. This is good, since not every distribution is a legal Wigner function.

There are some things that we can say about $e^{\Delta\tau\hat{T}}$ as $\Delta\tau \to \infty$. For example, if there is a similarity transformation that diagonalizes $\hat{T}$, the signs of the real parts of the eigenvalues are all we need to figure out the limiting forms. However, there should be no need to invent a procedure to figure out the limiting $\hat{T}$ and P_π's. Mathematicians should already have done this.

Someone has to start on this problem by searching the literature. For example, finding the limiting form, after an infinity of iterations, of the transition matrix in Markov processes is a very similar problem, though it may be worked out only for the discrete case.

Negative energies

Getting J_t from a general abstract $\hat{T}$ is all very well, but we also need to have a way of constructing transition matrices for physical systems defined by a Hamiltonian. To make things concrete, we can work with a simple harmonic oscillator, for which

$$H = \frac{1}{2}\left(x^2 + p^2\right) \tag{29}$$

To get backwards time travel, we need negative energies, just as $p < 0$ means a tendency to travel in the $-x$ direction.

How to implement this is a knotty question. It's possible that we may eventually have to extend phase space to include an energy axis paired with the time axis, but it is best if we can avoid that complication for now. We can still do some trial-and-error to get some insight into how to build transition matrices. For example, we can subtract a constant E from (29),

$$H = \frac{1}{2}\left(x^2 + p^2\right) - E \tag{30}$$

Classically, this is irrelevant, but in our case we can have particles tend to flow backwards in time when this modified $H < 0$.

Kevin Satzinger has written a computer program to help visualize currents; we may end up using that while playing around with such ideas.

Two-state system

I intend to use the simple harmonic oscillator as the system we should try to get this ideas to work in, mainly due to the symmetry between the roles of x and p in (29) and some other mathematical properties which makes it particularly well-behaved in standard approached to quantum mechanics.

But Wigner functions for a two-state system might be even simpler, and be described as a finite, discrete-indexed matrix. Since most mathematical literature on random walks and stochastic/Markov processes assumes finite, discrete-indexed situations, working with a two-state system and its Wigner functions might be a good idea.

Miguel Fernandez Flores has already found some literature on Wigner functions for two-state systems. So if the connection to the stochastic process literature is more straightforward, this would be worth exploring. One task would then be to translate much of this document to the case of a discrete phase space.