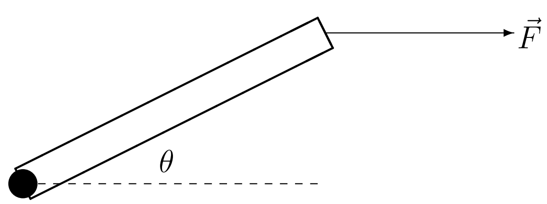
Solutions to Exam 2; Phys 185

1. (20 points) What would happen to the solar system the sun was suddenly replaced with a black hole with exactly the same mass as the sun?

Answer: If you replaced the Sun with a black hole that had exactly the same mass as the Sun (in other words, if you packed the mass of the Sun into a much smaller volume), you would get exactly the same gravitational forces, so all the planetary orbits would remain the same. So other than everything going dark, there would be no effect.

Gravity is gravity with black holes as well. Black holes are not cosmic vacuum cleaners that suck in everything around them, you can orbit them just like any mass.

2. (40 points) You have a rod of mass m and length l attached to an axis of rotation on one end as shown in the figure. A constant force $\vec{F}$—with unchanging magnitude and direction—is applied to the other end of the rod. You do this in outer space, where there are no other forces on the rod. The angle between the rod and a line parallel to the force is θ.

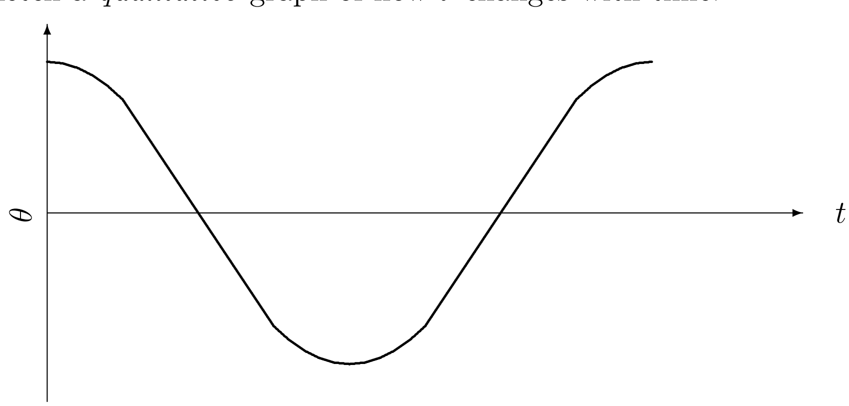


(a) Find an equation for the torque on the rod.

Answer: The perpendicular component of the force is $F_\perp = F\sin\theta$. The rotation seen in the diagram is clockwise. Therefore

$$\tau = -Fl\sin\theta$$

(b) Sketch a *qualitative* graph of how θ changes with time.



3. (30 points) You have a car traveling down the highway at constant speed v. Its wheels have a radius of 0.29 m and have 12 identical-looking spokes each, and they are rolling without slipping. You then film this car at a frame rate of 24 frames per second. Calculate the minimum speed v at which the tires will appear not to be rotating at all on film—so that at slightly smaller speeds the car wheels will appear to be rotating in the wrong direction.

Answer: If there are n spokes, we want the angle $\Delta\theta = 2\pi/n$ covered in time $\Delta t = 1/f$, where f is the frame rate. Then the angular velocity of the wheel is

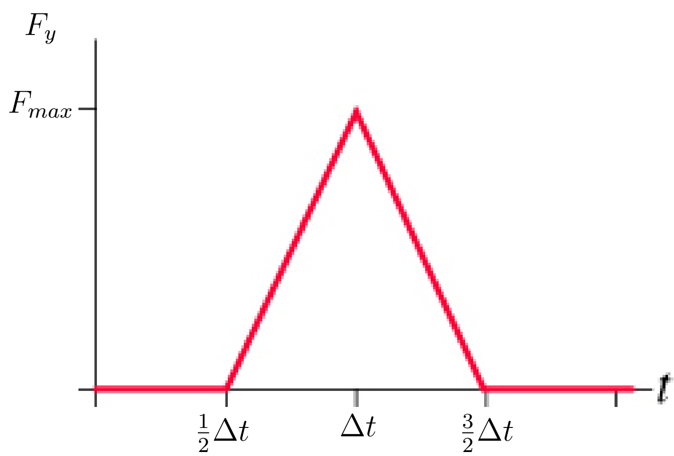
$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi f}{n}$$

With rolling without slipping,

$$v = \omega r = \frac{2\pi f r}{n} = 3.64 \text{ m/s} = 8.15 \text{ mph}$$

In other words, not very fast at all.

4. (40 points) A ball with mass m, starting at rest, is dropped from a height of h_i and bounces on a hard floor. The force on the ball from the floor is shown in the figure. Find the height h_f to which the ball rebounds.



Answer: First, the speed of the ball as it hits the floor. From energy conservation, $mgh_i = \frac{1}{2}mv_i^2$. Therefore $v_i = -\sqrt{2gh_i}$, with a $-$ sign because it is headed downward.

This means the momentum of the ball just before hitting the floor is $p_i = mv_i = -m\sqrt{2gh_i}$. The area under the curve is $\Delta p = p_f - p_i$, which is $\Delta p = \frac{1}{2}F_{max}\Delta t$. So on the bounce back up, the momentum starts from $p_f = p_i + \Delta p = -m\sqrt{2gh_i} + \frac{1}{2}F_{max}\Delta t$.

On the way up, we start with a velocity of $v_f = p_f/m = -\sqrt{2gh_i} + \frac{1}{2m}F_{max}\Delta t$. Using energy conservation again, with $\frac{1}{2}mv_f^2 = mgh_f$, we get

$$h_f = \frac{1}{2g}\left(\frac{F_{max}\Delta t}{2m} - \sqrt{2gh_i}\right)^2$$

5. (70 points) You have a solid planet (no oceans), with mass m_p and radius R_p, and its moon, with mass $m_m \ll m_p$ (the moon's mass is very small compared to the planet) and radius R_m. The orbit of the moon decays very gradually over time, with the moon slowly coming closer to the planet. Say the moon starts out at a circular orbit a distance r_i from the planet, and ten million years later, it ends up closer, on a circular orbit with radius $r_f < r_i$. We also have $R_m \ll r_f$, which means that you can treat the moon as a point particle when figuring out its moment of inertia, and that you can neglect the kinetic energy and angular momentum due to the spin of the moon about its own axis.

(a) Since gravity is the only force on the moon, it provides the centripetal acceleration. Find the initial and final angular velocities of the moon in orbit, ω_i and ω_f, in terms of the gravitational constant G, m_p, and r_i or r_f. (*Derive* this; don't just write down an equation from your cheat sheet!)

Answer: $\sum \vec{F} = m\vec{a}$ for uniform circular motion gives

$$G\frac{m_p m_m}{r^2} = m_m \omega^2 r \quad \Rightarrow \quad \omega = \sqrt{\frac{Gm_p}{r^3}} \quad \Rightarrow \quad \omega_i = \sqrt{\frac{Gm_p}{r_i^3}}, \quad \omega_f = \sqrt{\frac{Gm_p}{r_f^3}}$$

(b) As the orbit changes, the kinetic energy of the moon due to its orbital rotation changes. Find the moon's change in kinetic energy, $\Delta K_m = K_{mf} - K_{mi}$.

Answer: Since $K_m = \frac{1}{2} I_m \omega^2$ and $I_m = m_m r^2$,

$$\Delta K_m = K_{mf} - K_{mi} = \frac{1}{2}\frac{Gm_p}{r_f^3} m_m r_f^2 - \frac{1}{2}\frac{Gm_p}{r_i^3} m_m r_i^2 = \frac{Gm_p m_m}{2}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

(c) The general expression for gravitational potential energy is $U = -Gm_p m_m / r$. Use this to find the change in the moon's potential energy, $\Delta U_m = U_{mf} - U_{mi}$.

Answer:

$$\Delta U_m = U_{mf} - U_{mi} = -\frac{Gm_p m_m}{r_f} + \frac{Gm_p m_m}{r_i} = Gm_p m_m \left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

(d) Find the change in the moon's mechanical energy, $\Delta E_m = \Delta K_m + \Delta U_m$. Is this zero, positive, or negative?

Answer:

$$\Delta E_m = \Delta K_m + \Delta U_m = \frac{Gm_p m_m}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right) < 0$$

(e) The total angular momentum of the moon and planet system consists of the orbital angular momentum of the moon, and the angular momentum due to the rotation of the planet around its own axis. The moment of inertia for the planet is $\frac{2}{5}m_p R_p^2$. To keep the algebra simple, say the planet spins with an initial angular velocity $\Omega_i = 0$. Use angular momentum conservation to find the final angular velocity at which the planet spins, Ω_f.

Answer: The total angular momentum is the addition of the orbital angular momentum of the moon and the spin angular momentum of the planet. This is conserved:

$$I_{mi}\omega_i + I_p\Omega_i = I_{mf}\omega_f + I_p\Omega_f$$

$$m_m r_i^2\sqrt{\frac{Gm_p}{r_i^3}} + 0 = m_m r_f^2\sqrt{\frac{Gm_p}{r_f^3}} + \frac{2}{5}m_p R_p^2\Omega_f$$

Solving,

$$\Omega_f = \frac{5}{2}\frac{m_m}{R_p^2}\sqrt{\frac{G}{m_p}}\left(\sqrt{r_i} - \sqrt{r_f}\right)$$

(f) **(Bonus +10 points)** Find the change in the planet's kinetic energy due to its changing spin, $\Delta K_p = K_{pf} - K_{pi}$. Then write down the total change in mechanical energy of the whole system, $\Delta E = \Delta E_m + \Delta K_p$. Do you expect ΔE to be zero, positive, or negative?

Answer:

$$\Delta K_p = K_{pf} - K_{pi} = \frac{1}{2}I_p\Omega_f^2 - 0 = \frac{5Gm_m^2}{4R_p^2}\left(r_i + r_f - 2\sqrt{r_i r_f}\right)$$

$$\Delta E = \frac{Gm_p m_m}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right) + \frac{5Gm_m^2}{4R_p^2}\left(r_i + r_f - 2\sqrt{r_i r_f}\right)$$

You should expect $\Delta E < 0$, as $\Delta E_{th} = -\Delta E > 0$.