Solutions to Exam 2; Phys 185

1. (40 points) Why are brakes on a bicycle applied to the rim of a wheel and not on the axle? Model the bicycle wheel as a disk. Apply the same normal force with the brake pad, with the brake pad placed either near the rim or near the axle. The materials that come into contact are the same in both cases. In which case would the wheel come to rest faster? Construct a full argument, using equations as appropriate.

Answer: Stopping faster requires a higher rate of change of angular velocity: a large angular acceleration α. The disk's moment of inertia I does not depend on where the brake is applied. Therefore $\sum \tau = I\alpha$ means that the better location of the brake is that which produces a higher torque τ, which will result in a higher α slowing down. The force producing τ is kinetic friction $f_k = \mu_k n$. n is the same for both cases, and since the materials that come into contact are the same, then so is μ_k. Friction is perpendicular to the line between the point of application and the axis of rotation in both cases. So the only difference is the distance of the brake application point to the axis of the wheel. The larger torque will be produced by the brake that is farther away, since $\tau = f_k r$. Therefore the brake on the rim—with the larger r—will stop the wheel faster.

Alternatively: use energy. Stopping the wheel means converting all its rotational kinetic energy $\frac{1}{2}I\omega^2$ into ΔE_{th}. This thermal energy comes from the work done by friction at the brakes; $\Delta E_{th} = f_k \Delta x$, where Δx is the distance on the wheel over which friction acts. This distance is $\Delta x = r\Delta\theta$, with the change in angle in radians. You need the same Δx in either case, as the kinetic energy to dissipate is the same. The faster stop happens in the case where you have the smaller $\Delta\theta$ before you stop—that corresponds to the larger α. This means that you have to have a larger r—the rim.

2. (40 points) You have two masses m_1 and m_2, separated by a distance d, in outer space far from anything else. They both revolve, with the same angular velocity ω, about their common center of mass, which is stationary. Take the positions to be $x_1 = y_1 = y_2 = 0$ and $x_2 = d$.

(a) Show that in this case, the total linear momentum is zero and remains zero. (In other words, if the masses rotate about any point other than the center of mass, momentum won't be conserved.) *Hint:* For uniform circular motion, $v = \omega r$.

Answer: Take $x_1 = 0$, which means $x_2 = d$. Therefore the center of mass is at

$$x_{cm} = \frac{m_2 d}{m_1 + m_2}$$

The speed at which each mass moves at this instant is $v = \omega r$, where r is the distance to the center of mass. The velocities are perpendicular to the x-axis, and in opposite directions. So

$$v_{1y} = (x_1 - x_{cm})\omega = -\frac{m_2 d}{m_1 + m_2}\omega \qquad v_{2y} = (x_2 - x_{cm})\omega = \frac{m_1 d}{m_1 + m_2}\omega$$

The total momentum is then

$$p_{1y} + p_{2y} = m_1 v_{1y} + m_2 v_{2y} = -\frac{m_1 m_2 d}{m_1 + m_2}\omega + \frac{m_2 m_1 d}{m_1 + m_2}\omega = 0$$

which is what it should be when the center of mass is stationary. If the bodies rotate around any other point, their momenta will not add to zero, which would be inconsistent.

(b) Now assume that $m_1 \gg m_2$, so that $m_2/(m_1 + m_2) \approx 0$. (This is like a black hole and a small star.) Find an equation for m_1 in terms of d, ω, and the gravitational constant G.

Answer: If $m_2/(m_1 + m_2) \approx 0$, then $x_{cm} \approx 0$. The center of mass is almost at the first mass location. This means to a very good approximation, the second mass revolves around the first, with orbital radius $r \approx d$. Now we can set the total force, which is the gravitational attraction, equal to the force needed to produce the centripetal acceleration for uniform circular motion.

$$G\frac{m_1 m_2}{d^2} = m_2 \frac{v^2}{r}$$

Use $v = \omega r$ and $r \approx d$, to get

$$G\frac{m_1 m_2}{d^2} \approx m_2 \omega^2 d$$

Therefore,

$$m_1 \approx \frac{\omega^2 d^3}{G}$$

3. (40 points) Use the astronomical data in the back flap of your textbook to answer the following:

(a) About where is the axis of rotation of our solar system located? Is this exact, or an approximation? Explain.

Answer: This is very like the previous problem: The mass of the Sun is much larger than all the other planets: over a thousand Jupiters. They all rotate about their common center of mass, but the center of mass is almost at the center of the Sun.

(b) Calculate the moments of inertia of the Sun, Earth, and Jupiter. Be explicit about what approximations you are using to get the moment of inertia for each.

Answer: For the sun, use the moment of inertia for a sphere spinning on an axis through its own center; $I = \frac{2}{5}MR^2$ from the table in your textbook. For all the planets, you will notice from the table of astronomical data that their distances to the Sun are all much larger than their radii. Therefore the point particle approximation will work for them.

$$I_S = \frac{2}{5}(1.99 \times 10^{30}\ \text{kg})(6.96 \times 10^{8}\ \text{m})^2 = 3.86 \times 10^{47}\ \text{kg} \cdot \text{m}^2$$
$$I_E = (5.98 \times 10^{24}\ \text{kg})(1.50 \times 10^{11}\ \text{m})^2 = 1.35 \times 10^{47}\ \text{kg} \cdot \text{m}^2$$
$$I_J = (1.90 \times 10^{27}\ \text{kg})(7.78 \times 10^{11}\ \text{m})^2 = 1.15 \times 10^{51}\ \text{kg} \cdot \text{m}^2$$

(c) Calculate the angular momentum of the Sun, Earth, and Jupiter. (The Sun's period of rotation around its own axis is 24.5 days.) Add all these angular momenta together to get a total, and state what percentage of this total is associated with each.

Answer: Use $\omega = 2\pi/T$, where T is the period, and $L = I\omega$. 1 year is 3.15×10^7 seconds.

$$L_S = (3.86 \times 10^{47}\ \text{kg} \cdot \text{m}^2)\frac{2\pi}{(2.12 \times 10^6\ \text{s})} = 1.14 \times 10^{42}\ \text{kg} \cdot \text{m}^2/\text{s}$$

$$L_E = (1.35 \times 10^{47}\ \text{kg} \cdot \text{m}^2)\frac{2\pi}{(3.15 \times 10^7\ \text{s})} = 2.68 \times 10^{40}\ \text{kg} \cdot \text{m}^2/\text{s}$$

$$L_J = (1.15 \times 10^{51}\ \text{kg} \cdot \text{m}^2)\frac{2\pi}{(11.9)(3.15 \times 10^7\ \text{s})} = 1.93 \times 10^{43}\ \text{kg} \cdot \text{m}^2/\text{s}$$

The total is $L = 2.05 \times 10^{43}$ kg · m²/s. 5.7% of this belongs to the Sun, 0.13% to Earth, 94% to Jupiter. Most of the angular momentum of the solar system is in the rotation of Jupiter.

4. (40 points) Find v_{min}, the minimum speed a roller coaster must have at the bottom of a circular loop, if it is to make it all the way around a loop with radius r. Assume the track is frictionless.

Answer: There are no dissipative forces such as friction, so we can use energy conservation with $\Delta E_{th} = 0$. At the top of the loop, the height of the roller coaster is $2r$.

$$\tfrac{1}{2}mv_{min}^2 = \tfrac{1}{2}mv_{top}^2 + mg(2r)$$

For the minimum speed, at the top, the roller coaster must be barely in contact with the track, so that $n = 0$. Therefore the centripetal acceleration is provided entirely by gravity. On the y-axis, with down as the direction toward the center of the circle, $\sum F_y = ma_y$ gives

$$-mg = -m\frac{v_{top}^2}{r}$$

Therefore $v_{top}^2 = gr$. Putting that into the energy conservation equation,

$$\tfrac{1}{2}mv_{min}^2 = \tfrac{1}{2}mgr + mg(2r) = \tfrac{5}{2}mgr$$

$$v_{min} = \sqrt{5gr}$$

5. (40 points) The Earth is in orbit around the sun, and you can calculate its speed v_E at any moment. A comet with mass $m = 0.001m_E$ and speed v but going in the exact opposite direction approaches the Earth.

(a) Say the comet collides head-on with the Earth. Assume no significant material gets ejected into space. What must v be to produce a 1% change in the speed of the Earth?

Answer: This is a perfectly inelastic collision. Using momentum conservation,

$$m_E v_E - 0.001 m_E v = (m_E + 0.001 m_E)(0.99 v_E)$$

Solving,

$$v = [1 - (1.001)(0.99)]\frac{v_E}{0.001} = 9.01 v_E$$

The speed of the Earth is $2\pi r/T$, where r is the radius of the Earth's orbit, and T is 1 year.

$$v_E = \frac{2\pi(1.50 \times 10^{11}\ \text{m})}{(3.15 \times 10^7\ \text{s})} = 2.99 \times 10^4\ \text{m/s}$$

Therefore

$$v = 2.70 \times 10^5\ \text{m/s}$$

(b) Say the comet misses the Earth entirely, but interacts with Earth's gravity so that it gets a "slingshot" so that it ends up going straight back in the direction it came from. If there is, again, a 1% reduction in the speed of the Earth, what must the initial and final speeds of the comet be?

Answer: This is now a perfectly elastic collision, since there are no dissipative forces like friction involved. Now we have

$$m_E v_E - 0.001 m_E v_i = m_E(0.99 v_E) + 0.001 m_E v_f$$

$$\tfrac{1}{2} m_E v_E^2 + \tfrac{1}{2} 0.001 m_E v_i^2 = \tfrac{1}{2} m_E (0.99 v_E)^2 + \tfrac{1}{2} 0.001 m_E v_f^2$$

With some cancellations,

$$v_E - 0.001 v_i = 0.99 v_E + 0.001 v_f$$

$$v_E^2 + 0.001 v_i^2 = (0.99 v_E)^2 + 0.001 v_f^2$$

The first equation is easier: we get $v_f = 10 v_E - v_i$. Substituting that in the second,

$$1000\left(1 - 0.99^2\right) v_E^2 + v_i^2 = (10 v_E - v_i)^2 = 100 v_E^2 - 20 v_E v_i + v_i^2$$

Therefore

$$v_i = \frac{100 - 1000\left(1 - 0.99^2\right) v_E}{20} = 4.00 v_E = 1.20 \times 10^5 \text{ m/s}$$

and

$$v_f = 10 v_E - v_i = 6.00 v_E = 1.79 \times 10^5 \text{ m/s}$$