Solutions to Exam 3; Phys 185

1. (40 points) You have a hook fixed to the wall by a suction cup, on which you hang a towel of mass m. The radius of the suction cup is r. When you attach it to the wall, you push out the air between the wall and cup, and when it slightly expands again, the pressure inside the cup is $0.1p_{\text{atm}}$, and the outside pressure is p_{atm}. The coefficient of kinetic friction between the wall and the suction cup is μ_k, and and the coefficient of static friction is μ_s.

(a) Find an equation for the *maximum* mass m_{max} that can be hung on your hook.

Answer: The forces on the hook are: the weight of the towel w, the upward static friction force (at its maximum value) f_s, the force due to the pressure inside F_{in}—directed away from the wall, the force due to the external pressure F_{out}—directed toward the wall, and the normal force due to contact with the wall n. We want to hook not to move, so $a_x = a_y = 0$. Therefore

$$\sum F_x = n + F_{\text{in}} + F_{\text{out}} = 0$$

$$\sum F_y = f_s - w = 0$$

The forces due to the pressures are $F = pA$, where $A = \pi r^2$ is the area of the suction cup. Therefore, the x-equation gives

$$n = p_{\text{atm}}\pi r^2 - 0.1p_{\text{atm}}\pi r^2 = 0.9p_{\text{atm}}\pi r^2$$

With $f_{s,\text{max}} = \mu_s n$, the y-equation results in

$$m_{\text{max}}g = \mu_s 0.9p_{\text{atm}}\pi r^2$$

And therefore

$$m_{\text{max}} = \frac{0.9\mu_s p_{\text{atm}}\pi r^2}{g}$$

(b) Choose reasonable values for any variables you need and calculate m_{max}.

Answer: Take, for example, $p_{\text{atm}} = 1.01 \times 10^5$ Pa, $\mu_s = 0.5$, $r = 0.02$ m, $g = 9.8$ m/s^2. Then you get $m_{\text{max}} = 5.8$ kg, which is reasonable for a hook that holds towels.

2. (40 points) The most general equation for gravitational potential energy for two masses with separation r between their centers of mass is $U = -Gm_1m_2/r$. Note that U is negative, and that as r becomes very large, U approaches zero.

(a) If the total energy, $E = K + U$, of two gravitationally interacting objects is negative, they are *bound*: they will be in orbit around a common center of mass. But if $E \geq 0$, they are *unbound*: the separation between them can become arbitrarily large. Explain why this is so. *Hint 1:* Can K be negative? *Hint 2:* Look at very large r.

Answer: Since $K = \frac{1}{2}mv^2$, and since $m \geq 0$ and $v^2 \geq 0$, $K \geq 0$. As r becomes large, U increases to get closer to zero. Since energy is conserved $E = K + U$ must remain constant. Therefore, a larger U must be compensated for by a smaller K. But at really large r, $U \approx 0$. If $E \geq 0$, there is no problem, $E \approx K \geq 0$. But for negative energies, this is impossible: $E \approx K \geq 0$ contradicts $E < 0$. So r cannot get arbitrarily large with $E < 0$: a bound state.

(b) Using the astronomical data in your book, calculate the total energy for the Sun-Earth system and the Sun-Jupiter system. (Add the planets' kinetic energies due to their rotation around the Sun and their potential energies due to their distance from the Sun.) Do your results make sense in light of part (a)?

Answer: You can find data for r, the radius of the planets' orbits, and T, the periods. Then use

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{2\pi r}{T}\right)^2 \quad \text{and} \quad U = -G\frac{m m_{\text{sun}}}{r}$$

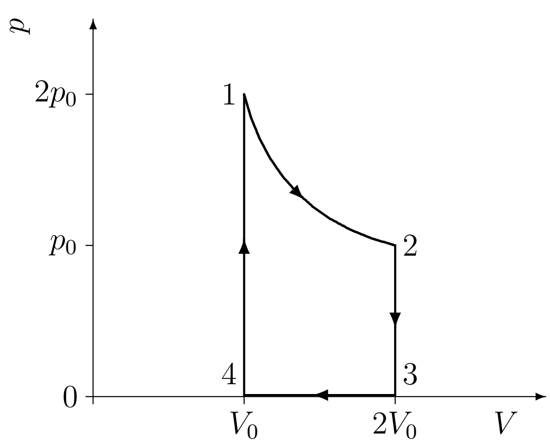
You get, for the Earth, $K_E = 2.67 \times 10^{33}$ J, $U_E = -5.29 \times 10^{33}$ J. For Jupiter, $K_J = 1.61 \times 10^{35}$ J, $U_J = -3.24 \times 10^{35}$ J. (Just for curiosity: notice that $U \approx -2K$ in either case. The relationship would have been exact if the planets had perfectly circular orbits.) Adding them up,

$$E_E = -2.62 \times 10^{33} \text{ J} < 0 \qquad E_J = -1.63 \times 10^{33} \text{ J} < 0$$

Both are negative, as they should be for planets bound to the Sun.

3. (40 points) It's physically impossible to have a cold reservoir at absolute zero, but let's see what would happen if such a thing were available.

You have a monatomic ideal gas that goes through the cycle $1 \to 2 \to 3 \to 4 \to 1$ shown in the diagram. No gas molecules are added or removed during the cycle.



Find everything (W, ΔE, Q) in terms of p_0 and V_0.

(a) The $1 \to 2$ part of the cycle takes place at *constant temperature*, so $T_1 = T_2$. The area under a constant temperature curve with temperature T on the p-V diagram, going from from an initial V_i to a final V_f, is

$$nRT \ln\left(\frac{V_f}{V_i}\right)$$

Find the work done by the gas for each step of this cycle: $W_{1\to2}$, $W_{2\to3}$, $W_{3\to4}$, $W_{4\to1}$.

Answer: Just take the areas under the curve, and note that $pV = nRT$

$$W_{1\to2} = nRT \ln\left(\frac{2V_0}{V_0}\right) = (2\ln 2)p_0V_0$$

All the other areas are zero, so $W_{2\to3} = W_{3\to4} = W_{4\to1} = 0$.

(b) Find the change in thermal energy for each step: $\Delta E_{1\to2}$, $\Delta E_{2\to3}$, $\Delta E_{3\to4}$, $\Delta E_{4\to1}$.

Answer: For a monatomic ideal gas, $\Delta E = E_f - E_i = \frac{3}{2}nRT_f - \frac{3}{2}nRT_i = \frac{3}{2}(p_fV_f - p_iV_i$. Therefore,

$$\Delta E_{1\to2} = \frac{3}{2}(2p_0V_0 - p_02V_0) = 0 \qquad \Delta E_{2\to3} = \frac{3}{2}(0 - p_02V_0) = -3p_0V_0$$

$$\Delta E_{3\to4} = \frac{3}{2}(0 - 0) = 0 \qquad \Delta E_{4\to1} = \frac{3}{2}(2p_0V_0 - 0) = 3p_0V_0$$

Notice that the ΔE's all sum to 0, which is as it should be a for a cycle.

(c) Find the heat added to the gas for each step of this cycle: $Q_{1\to2}$, $Q_{2\to3}$, $Q_{3\to4}$, $Q_{4\to1}$.

Answer: Use $Q = W + \Delta E$:

$$Q_{1\to2} = W_{1\to2} + \Delta E_{1\to2} = (2\ln 2)p_0V_0 \qquad Q_{2\to3} = W_{2\to3} + \Delta E_{2\to3} = -3p_0V_0$$

$$Q_{3\to4} = W_{3\to4} + \Delta E_{3\to4} = 0 \qquad Q_{4\to1} = W_{4\to1} + \Delta E_{4\to1} = 3p_0V_0$$

The sum of all Q's is equal to the total W, which is also as it should be.

(d) Find the total heat input to this gas in one cycle, Q_{in}. Also find the total heat removed from the gas, Q_{out}, and the total work done, W.

Answer: The total input is the sum of all positive Q steps:

$$Q_{\text{in}} = Q_{1\to2} + Q_{4\to1} = [3 + 2(\ln 2)]p_0V_0$$

The output is all the negative Q's:

$$Q_{\text{out}} = -Q_{2\to 3} = 3p_0V_0$$

The total work is

$$W = (2\ln 2)p_0V_0$$

Notice that $Q_{\text{in}} = Q_{\text{out}} + W$, as it should be for a heat engine.

(e) What is the efficiency of this heat engine? (Your result should be a number.)

Answer:

$$e = \frac{W}{Q_{\text{in}}} = \frac{2(\ln 2)}{3 + 2(\ln 2)} = 0.31$$

4. (40 points) If you look up how convection works, you will find $Q/\Delta t = hA\Delta T$, where A is the surface area of an object, and h is a convection coefficient that depends on the material and its geometric shape. You know how conduction and radiation works.

(a) You have two cubes made of identical materials, in identical environments, at identical starting temperatures. Cube 1 has a side of length a, cube 2 has $2a$. Find the ratio of the rates at which each cube cools:

$$\frac{\left(\frac{\Delta T_1}{\Delta t}\right)}{\left(\frac{\Delta T_2}{\Delta t}\right)}$$

Note: ΔT refers to the temperature difference with the environment. ΔT_1 and ΔT_2 are *different*—they refer to the change in temperature over time of cubes 1 and 2.

Hint: Your final result should be a number, with no symbols. Cancel things!

Answer: Add up the heat losses for cube 1:

$$\frac{Q_1}{\Delta t} = \frac{kA_1\Delta T}{L} + hA_1\Delta T + e\sigma A_1\Delta T^4 = \left[\frac{k\Delta T}{L} + h\Delta T + e\sigma\Delta T^4\right]A_1$$

Notice that the factor in brackets depends on the material of the cube and its surrounding environment—not the size of the cube! Calling the bracketed factor α, we then have

$$\frac{Q_1}{\Delta t} = \alpha A_1 \qquad \text{and} \qquad \frac{Q_2}{\Delta t} = \alpha A_2$$

For the temperature change, we use

$$Q_1 = m_1c\Delta T_1 = \rho V_1 c\Delta T_1 = [\rho c]V_1\Delta T_1$$

where ρ is the density of the material, and V_1 is the volume of cube 1. Again, the bracketed factor depends only on the material, and therefore is the same for both cubes. Call it β. So

$$Q_1 = \beta V_1 \Delta T_1 \qquad \text{and} \qquad Q_2 = \beta V_2 \Delta T_2$$

Putting the heat loss and the temperature equations together, we get

$$\frac{\beta V_1 \Delta T_1}{\Delta t} = \alpha A_1$$

With some rearranging,

$$\frac{\Delta T_1}{\Delta t} = \frac{\alpha}{\beta}\frac{A_1}{V_1} \qquad \text{and} \qquad \frac{\Delta T_2}{\Delta t} = \frac{\alpha}{\beta}\frac{A_2}{V_2}$$

Finally, when we take the ratio, the constants α and β cancel out, and we're just left with the ratios of surface-area-to-volume ratios:

$$\frac{\left(\frac{\Delta T_1}{\Delta t}\right)}{\left(\frac{\Delta T_2}{\Delta t}\right)} = \frac{\frac{\alpha}{\beta}\frac{A_1}{V_1}}{\frac{\alpha}{\beta}\frac{A_2}{V_2}} = \frac{\frac{A_1}{V_1}}{\frac{A_2}{V_2}} = \frac{\frac{6a^2}{a^3}}{\frac{6(2a)^2}{(2a)^3}} = 2$$

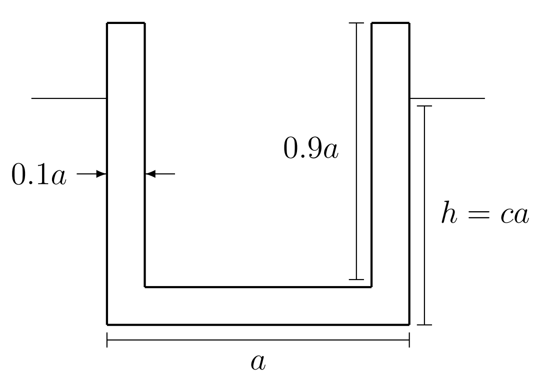
The small cube will cool down twice as fast.

(b) Use this to predict whether in cold climates, small or large animals will have proportionally thicker coats, and area-reducing adaptations such as smaller external ears. Explain.

Answer: The result of (a) means that smaller animals cool down faster. So they have higher adaptive pressure on them to reduce their heat loss—proportionally thicker coats, and smaller external ears and so forth.

5. (40 points) A plastic tub has a square base with sides a and thickness $0.100a$. Its four walls have a height of $0.900a$ and thickness $0.100a$. The plastic has density $\rho_p = 2.00\rho_w$, where ρ_w is the density of water. You set the tub to float in water. h, the distance from the bottom of the tub to the waterline, will be related to a like $h = ca$, where c is a number. Find c.

Hint: You have to be careful calculating the volume of the tub. Be sure to ask me to check whether you have that right.



Answer: The two forces on the tub are its weight and the buoyancy force. Since the tub floats, these forces must be equal and opposite. The mass of the tub is $\rho_p V_{\text{tub}}$, so

$$\rho_p V_{\text{tub}} g = \rho_w V_{\text{sub}} g$$

where V_{sub} is the submerged volume. Substituting $\rho_p = 2.00\rho_w$ and doing some cancellations,

$$2V_{\text{tub}} = V_{\text{sub}}$$

So we need the volumes. For the tub, subtract the hollow part from the full cube:

$$V_{\text{tub}} = a^3 - (0.8a)^2(0.9a) = 0.424\,a^3$$

The submerged volume $V_{\text{sub}} = a^2 h$. Putting these in the equation,

$$2(0.424\,a^3) = a^2 h \qquad \Rightarrow \qquad h = 0.848\,a$$

In other words, $c = 0.848$.