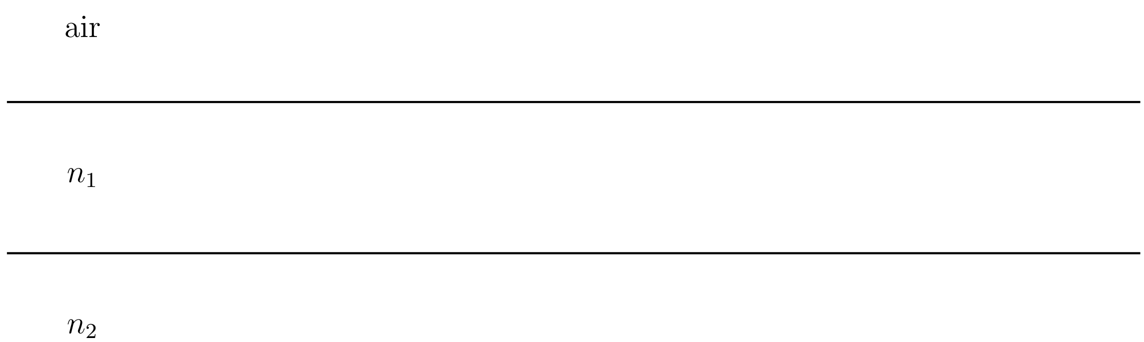
Solutions to Exam 1; Phys 186

1. (40 points) A layer of material 1 with index of refraction n_1 stands on a horizontal slab of material 2 with index of refraction n_2, with $n_2 > n_1$. Light from within the slab of material 2 is incident on the boundary between 1 and 2. Find an equation for the maximum angle of incidence for which a light ray can emerge into the air above layer 1.



Answer: The interface between material 1 and air gives

$$n_{\text{air}} \sin\theta_{\text{air}} = n_1 \sin\theta_1$$

Since we are looking for a maximum θ_2, θ_{air} must be at its maximum, so that $\sin\theta_{\text{air}} = 1$. Together with $n_{\text{air}} = 1$, this means

$$1 = n_1 \sin\theta_1$$

From the second interface we get

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

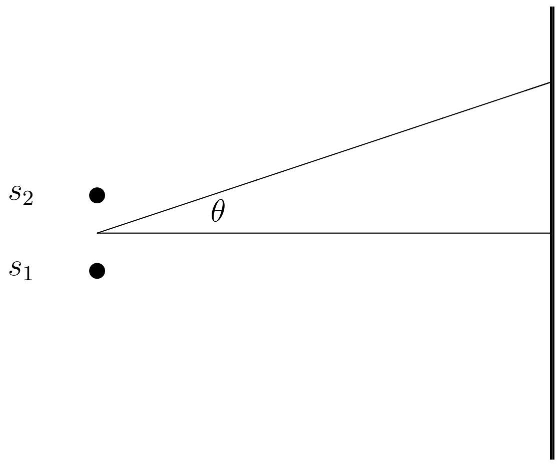
Therefore

$$1 = n_2 \sin\theta_2$$

and

$$\theta_2 = \sin^{-1}\left(\frac{1}{n_2}\right)$$

2. (40 points) You have two sources of waves with identical frequencies, amplitudes, and wavelengths, s_1 and s_2, separated by a distance of 2.2 wavelengths: $d = 2.2\lambda$. To the right there is a screen that records intensity at angles θ.



(a) s_1 and s_2 are completely in phase: they emit peaks and troughs of waves simultaneously. Plot a rough intensity versus angle graph. Be careful about whether your minima are exactly zero are not. Calculate and indicate the angles at which the *maxima* occur.

Answer: This is the exact same situation as a double-slit experiment. So the intensity graph will be a classic double-slit pattern: A central maximum peak, and other less-high maxima off-center. In between will be points of destructive interference—but these minima will not be at exactly zero since the distances from the screen to the sources will not be equal and hence the wave amplitudes will not be equal. The maxima location are expressed by

$$\sin\theta = m\frac{\lambda}{d}, \qquad m = 0, \pm 1, \pm 2, \ldots$$

Since $\lambda/d = 2.2$, we get maxima at

$$m = 0: \quad \theta = 0$$

$$m = \pm 1: \quad \theta = \sin^{-1}\left(\frac{\pm 1}{2.2}\right) = 27^\circ$$

$$m = \pm 2: \quad \theta = \sin^{-1}\left(\frac{\pm 2}{2.2}\right) = 65^\circ$$

Larger m values give no solutions, so there are only 5 peaks.

(b) s_1 and s_2 are completely out of phase: when one emits a peak the other emits a trough. Plot a rough intensity versus angle graph. Be careful about whether your minima are exactly zero are not. Indicate the angles at which the *minima* occur.

Answer: Interchanging peaks and troughs for one source just exchanges points of constructive and destructive interference. So the maxima calculated for (a) will now be the minima. Indeed, the center at $\theta = 0$ with have $I = 0$, since it is equidistant to both sources and the wave amplitudes are the same—canceling each out completely. The maxima will be at the locations of the previous minima, decreasing in height as the angle moves away from the center.

3. (30 points) You have a spring with stiffness (spring constant) k with a mass m attached. At time $t = 0$, you extend the spring by a distance d away from equilibrium, and let it go from rest. At the same time, you also have another spring with stiffness $2k$ with a mass of $m/2$ attached. At $t = 0$, you extend this second spring by $2d$ away from equilibrium, and let it go from rest.

On the graph below, sketch the position of the mass versus time for both springs, over *two periods* of the first spring. Show your calculations below the graph.

Answer: The curve for the first spring will be two periods of a cosine curve, with amplitude d. The second spring will also be a cosine curve, but with amplitude $2d$.

The frequency of the second spring will also differ. Since $\omega_1 = \sqrt{k/m}$,

$$\omega_2 = \sqrt{\frac{2k}{m/2}} = 2\sqrt{\frac{k}{m}} = 2\omega_1$$

Double the frequency means that in the time that spring 1 completes two periods, the second spring will complete four.

4. (40 points) Roughly sketch equipotential lines (solid lines) and electric field lines (dashed lines with arrows indicating direction) in the following situations.

(a) Two parallel plates with equal and opposite charges upon them.

Answer: This is a parallel plate capacitor, with a constant electric field. The electric field lines will be equally spaced, straight and parallel, and will go from the + plate to the − plate. The equipotential lines will be equally spaced and parallel to the plates, perpendicular to the field lines.

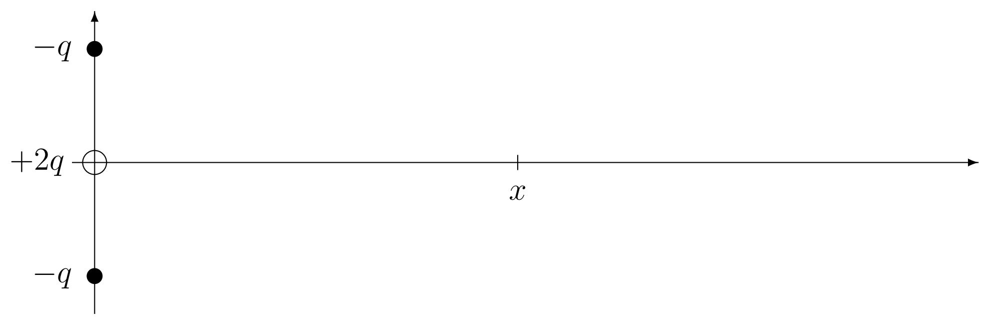
(b) Now say you placed a metal cylinder in between the plates. Inside the cylinder, roughly indicate how charges will be redistributed by placing + and − signs. Outside, roughly draw in the equipotential lines and electric field lines due to this induced charge distribution.

Answer: The free − charges (electrons) in the metal will be attracted by the + plate and migrate to the left side. The right side will therefore have an excess of + charge left. This is approximately a dipole charge configuration. So the electric field and equipotential lines will follow the the familiar dipole picture.

(c) Now draw in the equipotential lines and total electric field lines for the combined configuration: what is due to the plates *plus* the induced charge distribution on your cylinder. Include the lines within the cylinder. This is what you would see in the lab with your voltage probe.

Answer: This produces what you saw in the lab: the metal is an equipotential, and the equipotential lines between the plates are distorted to allow the cylinder in between. The electric field lines will bend toward the surface of the metal, ending perpendicular to the metal. There is no electric field inside.

5. (50 points) You have a quadrupole charge distribution with charges $-q$ at $x = 0, y = a$ and $x = 0, y = -a$, and a charge $+2q$ at $x = 0, y = 0$. The positions of the charges relative to each other are fixed.



(a) Find the total electric field due to the quadrupole, at a point a distance x from the origin on the x axis. In other words, find the total E_x and E_y as functions of k, q, a, and x.

Answer: Label the charges 1, 2, 3, from the top. The magnitudes of the electric fields are

$$E_1 = E_3 = \frac{kq}{a^2 + x^2} \qquad E_2 = \frac{2kq}{x^2}$$

From the symmetry of the set-up, you can see that the y-components of $\vec{E}_1$ and $\vec{E}_3$ will cancel each other out, leaving a total $E_y = 0$. The x-components are, when we include the factors of $\cos\theta = x/\sqrt{a^2 + x^2}$,

$$E_{1x} = E_{3x} = -\frac{kqx}{(a^2 + x^2)^{3/2}} \qquad E_{2x} = \frac{2kq}{x^2}$$

Adding all these up,

$$E_x = 2kq\left(\frac{1}{x^2} - \frac{x}{(a^2 + x^2)^{3/2}}\right)$$

(b) If you can cast your previous answer in a form that puts all the a-dependence in a term that goes like $(1 + a^2/x^2)^{-3/2}$, you can use the approximation

$$\left(1+\frac{a^2}{x^2}\right)^{-3/2} \approx 1-\frac{3a^2}{2x^2}$$

valid for $a \ll x$. Use this to show that for large x values, the electric field due to a quadrupole goes like $E \propto x^{-4}$, an inverse fourth power law.

Answer: Rewrite the previous answer as

$$E_x = \frac{2kq}{x^2}\left[1-\left(1+\frac{a^2}{x^2}\right)^{-3/2}\right]$$

Use the approximation:

$$E_x \approx \frac{2kq}{x^2}\left[1-\left(1-\frac{3a^2}{2x^2}\right)\right] = \frac{3kqa^2}{x^4} \propto x^{-4}$$