Homework Solutions #3 (McIntyre Chapter 3)

2 This is fairly trivial.

$$P_{E=E_n} = \left|c_n e^{-itE_n/\hbar}\right|^2 = |c_n|^2 \qquad \text{independent of } t$$

Other observables that commute with H will have a set of common eigenstates. So they will have t-independent probabilities for exactly the same reason.

6 Less trivially,

(a) After measurement,

$$|\psi(0)\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

(b) $H = \omega_0 S_z$ is diagonal in the $\{|+\rangle, |-\rangle\}$ basis. Therefore

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 T/2}|+\rangle + e^{i\omega_0 T/2}|-\rangle\right)$$

(c) Now use the $\{|+\rangle_y, |-\rangle_y\}$ basis, which makes the new $H = \omega_0 S_y$ diagonal.

$$|\psi(T)\rangle = |+\rangle_{yy}\langle+|\psi(T)\rangle + |-\rangle_{yy}\langle-|\psi(T)\rangle =$$
$$\frac{1}{2}\left(e^{-i\omega_0 T/2} - ie^{i\omega_0 T/2}\right)|+\rangle_y + \frac{1}{2}\left(e^{-i\omega_0 T/2} + ie^{i\omega_0 T/2}\right)|-\rangle_y$$

After another T passes,

$$|\psi(2T)\rangle = \frac{1}{2}\left(e^{-i\omega_0 T/2} - ie^{i\omega_0 T/2}\right)e^{-i\omega_0 T/2}|+\rangle_y +$$
$$\frac{1}{2}\left(e^{-i\omega_0 T/2} + ie^{i\omega_0 T/2}\right)e^{i\omega_0 T/2}|-\rangle_y$$

The probability is then

$$\mathcal{P}_{S_x=\hbar/2} = |{}_x\langle+|\psi(2T)\rangle|^2 = \frac{1}{2}(1+\cos^2\omega_0 T)$$

9 The unit vector $\hat{\mathbf{n}}$ corresponds to $\theta = \pi/2$ and $\phi = \pi/4$. Therefore, in the $\{|+\rangle, |-\rangle\}$ basis where $H = \omega_0 S_z$ is diagonal,

$$|\psi(0)\rangle = |+\rangle_n = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\pi/4}|-\rangle$$

The time evolution is then

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle + e^{i\omega_0 t/2}e^{i\pi/4}|-\rangle\right)$$

And the probability of interest is

$$\mathcal{P}_{S_y=\hbar/2} = \left|{}_y\langle+|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|e^{-i\omega_0 t/2} + e^{-i\pi/2}e^{i\omega_0 t/2}e^{i\pi/4}\right|^2 =$$

$$\frac{1}{4}\left|e^{-i\pi/8}\left(e^{-i\omega_0 t/2}e^{i\pi/8} + e^{i\omega_0 t/2}e^{-i\pi/8}\right)\right|^2 = \cos^2\left(\frac{\omega_0 t}{2} - \frac{\pi}{8}\right)$$

14 Normalization requires $C = \frac{1}{5}$.

(a) Solving the eigenvalue problem gives the eigenvalues $E_1 = E_0$ and $E_2 = 3E_0$, with

$$|E_1\rangle = \frac{1}{\sqrt{2}}\left(|a_1\rangle - |a_2\rangle\right) \qquad |E_2\rangle = \frac{1}{\sqrt{2}}\left(|a_1\rangle + |a_2\rangle\right)$$

Expressing the state in the basis where H is diagonal,

$$|\psi(0)\rangle = I|\psi(0)\rangle = |E_1\rangle\langle E_1|\psi(0)\rangle + |E_2\rangle\langle E_2|\psi(0)\rangle = \frac{1}{5\sqrt{2}}\left(-|E_1\rangle + 7|E_2\rangle\right)$$

With the time evolution,

$$|\psi(t)\rangle = \frac{1}{5\sqrt{2}}\left(-e^{-i\omega_1 t}|E_1\rangle + 7e^{-i\omega_2 t}|E_2\rangle\right)$$

where $\omega_1 = E_0/\hbar$ and $\omega_2 = 3E_0/\hbar$. The probabilities are then

$$\mathcal{P}_{E=E_1} = \left|\langle E_1|\psi(t)\rangle\right|^2 = \frac{1}{50} \qquad \mathcal{P}_{E=E_2} = \left|\langle E_2|\psi(t)\rangle\right|^2 = \frac{49}{50}$$

(b) The expectation value $\langle A \rangle$ is best obtained in the $\{|a_1\rangle, |a_2\rangle\}$ basis, because the action of A on these states is known. So start with expressing the state in this basis:

$$|\psi(t)\rangle = I|\psi(t)\rangle = |a_1\rangle\langle a_1|\psi(t)\rangle + |a_2\rangle\langle a_2|\psi(t)\rangle =$$

$$\frac{1}{10}\left[\left(-e^{-i\omega_1 t} + 7e^{-i\omega_2 t}\right)|a_1\rangle + \left(e^{-i\omega_1 t} + 7e^{-i\omega_2 t}\right)|a_2\rangle\right]$$

Using $A|a_1\rangle = a_1|a_1\rangle$ and $A|a_2\rangle = a_2|a_2\rangle$,

$$\langle\psi(t)|A|\psi(t)\rangle =$$

$$\frac{1}{100}\left[50 - 14\cos(\omega_2 - \omega_1)t\right]a_1 + \frac{1}{100}\left[50 + 14\cos(\omega_2 - \omega_1)t\right]a_2 =$$

$$\left[0.5 - 0.14\cos(2E_0 t/\hbar)\right]a_1 + \left[0.5 + 0.14\cos(2E_0 t/\hbar)\right]a_2$$