Solutions to First Midterm

1. (50 points) You've learned how to figure out the eigenvalues and eigenvectors of a matrix. Now let's do the opposite: construct a matrix given its eigenvalues and eigenvectors.

You have two vectors $|\phi_1\rangle$ and $|\phi_2\rangle$, where

$$\langle\phi_m|\phi_n\rangle = \delta_{mn} \qquad m = 1, 2 \quad n = 1, 2$$

You also have vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, such that

$$\langle\phi_1|\psi_1\rangle = \frac{5}{N} \quad \langle\phi_2|\psi_1\rangle = \frac{3-4i}{N} \qquad \langle\phi_1|\psi_2\rangle = \frac{5}{N} \quad \langle\phi_2|\psi_2\rangle = \frac{4i-3}{N}$$

In other words, in the ϕ-basis,

$$|\psi_1\rangle \doteq \frac{1}{N}\begin{pmatrix} 5 \\ 3-4i \end{pmatrix} \qquad |\psi_2\rangle \doteq \frac{1}{N}\begin{pmatrix} 5 \\ 4i-3 \end{pmatrix}$$

(a) Find the normalization factor $1/N$ needed to normalize $|\psi_1\rangle$ and $|\psi_2\rangle$. Then confirm that $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis set—check that

$$\langle\psi_m|\psi_n\rangle = \delta_{mn} \qquad m = 1, 2 \quad n = 1, 2$$

Answer: Normalization:

$$\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = \frac{1}{N^2}(25 + 25) = 1 \quad \Rightarrow \quad N = \sqrt{50}$$

Orthogonality:

$$\langle\psi_1|\psi_2\rangle = \frac{1}{N^2}(25 + 12i - 9 - 16 - 12i) = 0$$

So everything is fine.

(b) Say that $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenvectors of a linear operator $\hat{Q}$, with eigenvalues $+1$ and -1:

$$\hat{Q}|\psi_1\rangle = (+1)|\psi_1\rangle \qquad \hat{Q}|\psi_2\rangle = (-1)|\psi_2\rangle$$

Find the matrix for $\hat{Q}$ in the ϕ-basis. In other words, find the matrix elements to form

$$\hat{Q} \doteq \begin{pmatrix} \langle\phi_1|\hat{Q}|\phi_1\rangle & \langle\phi_1|\hat{Q}|\phi_2\rangle \\ \langle\phi_2|\hat{Q}|\phi_1\rangle & \langle\phi_2|\hat{Q}|\phi_2\rangle \end{pmatrix}$$

Answer: Using Dirac notation (the safe method):

$$\langle\phi_n|\hat{Q}|\phi_m\rangle = \langle\phi_n|\hat{I}\hat{Q}\hat{I}|\phi_n\rangle$$

where $\hat{I} = |\psi_1\rangle\langle\psi_1| + \psi_2\rangle\langle\psi_2|$. Therefore

$$\begin{aligned} \langle\phi_n|\hat{Q}|\phi_m\rangle &= \langle\phi_n|\psi_1\rangle\langle\psi_1|\hat{Q}|\psi_1\rangle\langle\psi_1|\phi_m\rangle + \langle\phi_n|\psi_1\rangle\langle\psi_1|\hat{Q}|\psi_2\rangle\langle\psi_2|\phi_m\rangle + \\ &\quad \langle\phi_n|\psi_2\rangle\langle\psi_2|\hat{Q}|\psi_1\rangle\langle\psi_1|\phi_m\rangle + \langle\phi_n|\psi_2\rangle\langle\psi_2|\hat{Q}|\psi_2\rangle\langle\psi_2|\phi_m\rangle \\ &= (+1)\langle\phi_n|\psi_1\rangle\langle\psi_1|\phi_m\rangle + 0 + 0 + (-1)\langle\phi_n|\psi_2\rangle\langle\psi_2|\phi_m\rangle \end{aligned}$$

Putting in the inner products given in the question, and remembering that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$, we get

$$\hat{Q} \doteq \frac{1}{50}\begin{pmatrix} 25-25 & 30+40i \\ 30-40i & 25-25 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 0 & 3+4i \\ 3-4i & 0 \end{pmatrix}$$

An alternative method is realizing that in the ψ-basis $\hat{Q}$ is diagonal, and so we can just use the transformation matrix made out of the eigenvectors to go back to the ϕ-basis.

$$\begin{aligned} \hat{Q} &\doteq \frac{1}{50}\begin{pmatrix} 5 & 5 \\ 3-4i & 4i-3 \end{pmatrix}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & 3+4i \\ 5 & -3-4i \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 0 & 3+4i \\ 3-4i & 0 \end{pmatrix} \end{aligned}$$

2. (20 points) You have a function of N variables

$$f(x_1, x_2, \ldots, x_N) = \sum_{n=1}^{N} n\, x_n^2$$

Using Lagrange multipliers, find the maximum value for f and the location of this maximum, subject to the constraint that

$$\sum_{n=1}^{N} x_n^2 = 1$$

Note: This is a trivial problem that doesn't technically need Lagrange multipliers. I just want to check whether you know how to use Lagrange multipliers with a problem where you don't have to do any long algebra.

Answer: Maximize

$$F = \sum_{n=1}^{N} n\, x_n^2 - \lambda \left(\sum_{n=1}^{N} x_n^2 - 1 \right)$$

Therefore, for all m,

$$\frac{\partial F}{\partial x_m} = 2m\, x_m - 2\lambda\, x_m = 0 \quad \Rightarrow \quad x_m(m - \lambda) = 0$$

The solutions are $x_m = 0$ or $\lambda = m$. All the x_m cannot be zero, otherwise the constraint is not satisfied. $\lambda = m$ can happen for only one value of m. Therefore $x_n = 0$ for all $n \neq m$, for a given m. And for that case, $x_m = \pm 1$, from the constraint equation.

The largest value for f will appear when m is the largest possible value: $m = N$. So in this case, the maximum is $f = N$, at $x_n = 0$ for $n < n$ and $x_N = \pm 1$.

3. (30 points) You have a vector potential $\mathbf{A} = B_0 x\, \mathbf{j}$, where B_0 is a constant. This leads to a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

(a) Find $\mathbf{B}$ and do the area integral for the magnetic flux,

$$\int_\sigma d\sigma\, \mathbf{n} \cdot \mathbf{B}$$

where the area σ is the circle $x^2 + y^2 \leq 4$ and $z = 0$.

Answer: The magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A} = B_0 \mathbf{k}$$

The normal vector $\mathbf{n} = \mathbf{k}$ as well, so

$$\int_\sigma d\sigma\, \mathbf{n} \cdot \mathbf{B} = B_0 \int_\sigma d\sigma = 4\pi B_0$$

where 4π is the area of the circle.

(b) Do the line integral

$$\int_{\partial\sigma} d\mathbf{r} \cdot \mathbf{A}$$

where $\partial\sigma$ is the boundary of the same circle.

Hint: Use polar coordinates, and notice that the unit vector in the direction of $d\mathbf{r}$ is $\mathbf{e}_\theta$ given in equation 4.12 in chapter 6.

Answer: In this case, with radius $r = 2$, $d\mathbf{r} = d\theta\, r\mathbf{e}_\theta = d\theta\, 2\mathbf{e}_\theta$, and $x = r\cos\theta = 2\cos\theta$.

$$\int_{\partial\sigma} d\mathbf{r} \cdot \mathbf{A} = 2B_0 \int_0^{2\pi} d\theta\, \cos\theta(2\cos\theta) = 4\pi B_0$$

The same, as it should be.