Homework Solutions, Chapter 3

6.6 All of this works out trivially if you do the appropriate matrix multiplications.

6.30 We know that $\hat{A}^n = \hat{I}$ for even n, and $\hat{A}^n = \hat{A}$ for odd n. Therefore

$$\sin k\hat{A} = \sum_{n=0}^{\infty}(-1)^n \frac{(k\hat{A})^{2n+1}}{(2n+1)!} = \hat{A}\sum_{n=0}^{\infty}(-1)^n \frac{k^{2n+1}}{(2n+1)!} = \hat{A}\sin k$$

$$\cos k\hat{A} = \sum_{n=0}^{\infty}(-1)^n \frac{(k\hat{A})^{2n}}{(2n)!} = \hat{I}\sum_{n=0}^{\infty}(-1)^n \frac{k^{2n}}{(2n)!} = \hat{I}\cos k$$

$$e^{ik\hat{A}} = \cos k\hat{A} + i\sin k\hat{A} = \hat{I}\cos k + \hat{A}i\sin k = \begin{pmatrix} \cos k & i\sin k \\ i\sin k & \cos k \end{pmatrix}$$

$$e^{k\hat{A}} = \hat{I}\cosh k + \hat{A}\sinh k = \begin{pmatrix} \cosh k & \sinh k \\ \sinh k & \cosh k \end{pmatrix}$$

7.29 We want a matrix that will do the transformation $A_x \to A_z$, $A_y \to -A_y$, and $A_z \to -A_x$. The matrix that does this is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

8.10 Do the Wronskian bit:

$$\begin{vmatrix} x & e^x & x\,e^x \\ 1 & e^x & e^x + x\,e^x \\ 0 & e^x & 2e^x + x\,e^x \end{vmatrix} = e^{2x}(x-2) \neq 0$$

Therefore, linearly independent.

9.23 Using basic identities,

$$(\hat{A}\hat{A}^\dagger)^\dagger = \hat{A}^{\dagger\dagger}\hat{A}^\dagger = \hat{A}\hat{A}^\dagger$$

$$(\hat{A}+\hat{A}^\dagger)^\dagger = \hat{A}^\dagger + \hat{A}^{\dagger\dagger} = \hat{A}+\hat{A}^\dagger$$

$$\left[i(\hat{A}-\hat{A}^\dagger)\right]^\dagger = -i(\hat{A}^\dagger - \hat{A}^{\dagger\dagger}) = i(\hat{A}-\hat{A}^\dagger)$$

11.42 First, eigenvalues. Solve

$$\begin{vmatrix} 3-\lambda & 1-i \\ 1+i & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 4 = 0$$

Using the quadratic formula, giving $\lambda_1 = 1$ and $\lambda_2 = 4$. The normalized eigenvectors corresponding to these are

$$\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1+i \end{pmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

The unitary matrix and the diagonalized H is

$$\hat{U} = \frac{1}{\sqrt{6}}\begin{pmatrix} -\sqrt{2} & 2 \\ \sqrt{2}(1+i) & 1+i \end{pmatrix} \qquad U^\dagger \hat{H} U = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

14.2 Let two vectors be $\mathbf{A} = a_1 e^x + a_2 \sinh x + a_3 x\, e^x$ and $\mathbf{B} = b_1 e^x + b_2 \sinh x + b_3 x\, e^x$.

1. $\mathbf{A}+\mathbf{B}$ is also a vector.
2. $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$, plus $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$.
3. The zero vector is the one where $a_1 = a_2 = a_3 = 0$, and $-\mathbf{A}$ just has $a_1 = -a_1$ and so forth.
4. You can see that $k(\mathbf{A}+\mathbf{B}) = k\mathbf{A}+k\mathbf{B}$, etc.

The basis is $\{e^x,\, \sinh x,\, x\, e^x\}$, and this is a 3D space.

14.11 If $\langle A|B\rangle = \int dx\, A^* B$, then

$$\langle A|B\rangle^* = \int dx\, B^* A = \langle B|A\rangle$$

$$\langle A|A\rangle = \int dx\, A^* A = \int dx\, |A|^2 \geq 0 \quad \text{and} \quad \langle A|A\rangle = 0 \Leftrightarrow A = 0$$

$$\begin{aligned}\langle C|aA+bB\rangle &= \int dx\, C^*(aA+bB)\\ &= a\int dx\, C^*A + b\int dx, C^*B = a\langle C|A\rangle + b\langle C|B\rangle\end{aligned}$$

$$\begin{aligned}\langle aA+bB|C\rangle &= \int dx\, (aA+bB)^*C\\ &= a^*\int dx\, A^*C + b^*\int dx, B^*C = a^*\langle A|C\rangle + b^*\langle B|C\rangle\end{aligned}$$

$$\langle aA|bB\rangle = \int dx\, a^*A^*bB = a*b\int dx\, A^*B = a^*b\langle A|B\rangle$$