Homework Solutions, Chapter 4

1.18 Using $r^2 = x^2 + y^2$, we can write z as a function of y and r *only*: $z = (x^2 + y^2) + y^2 = r^2 + y^2$. So

$$\left(\frac{\partial z}{\partial r}\right)_y = 2r$$

5.7 Straightforward application of chain rule:

$$\frac{dc}{da} = \frac{\partial c}{\partial a} + \frac{\partial c}{\partial b}\frac{db}{da}$$

With $db/da = (1+2a)e^{2a}$, $\partial c/\partial a = \cos(a-b)$, and $\partial c/\partial b = -\cos(a-b)$, we get

$$\frac{dc}{da} = \cos(a - a\,e^{2a})\left[1-(1+2a)e^{2a}\right]$$

7.23 Call $a = x - ct$ and $b = x + ct$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial x} = \frac{\partial f}{\partial a} + \frac{\partial g}{\partial b}$$

Differentiating again gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial a^2} + \frac{\partial^2 g}{\partial b^2}$$

The t-derivatives work the same way, with

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial t} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial t} = -c\frac{\partial f}{\partial a} + c\frac{\partial g}{\partial b}$$

The second derivative gives a c^2 in each term:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 f}{\partial a^2} + c^2\frac{\partial^2 g}{\partial b^2}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

9.2 The volume and area are:

$$V(r,l,s) = \pi r^2 \left(l + \frac{1}{3}\sqrt{s^2 - r^2} \right)$$

$$A(r,l,s) = \pi r(r + 2l + s)$$

Maximizing $f = V - \lambda A$ gives

$$\frac{\partial f}{\partial r} = 2\pi r \left(l + \frac{1}{3}\sqrt{s^2 - r^2} \right) - \frac{\pi r^3}{3\sqrt{s^2 - r^2}} - 2\lambda\pi r - 2\lambda\pi l - \lambda\pi s = 0 \quad (1)$$

$$\frac{\partial f}{\partial l} = \pi r^2 - 2\lambda\pi r = 0 \quad (2)$$

$$\frac{\partial f}{\partial s} = \frac{\pi s r^2}{3\sqrt{s^2 - r^2}} - \lambda\pi r = 0 \quad (3)$$

Equation (2) gives $\lambda = r/2$. Replacing this in equation (3) results in $r = cs$, where $c = \sqrt{5}/3 = 0.745$.

Putting all this in equation (1) results in

$$l = \left(\frac{1}{2} + c + \frac{c^2}{3k} - \frac{2k}{3} \right) s$$

where $k = \sqrt{1 - c^2} = 2/3$. Hence $l = 1.079\, s$.

10.11 $\partial T/\partial x = 4x - 2 = 0$ for $x = 0.5$; and $\partial T/\partial y = -6y = 0$ giving $y = 0$. $T(0.5, 0) = 9.5$ as the extremum inside.

Now check the boundary. At $y = \pm 1$, the extremum is at $x = 0.5$ again, so $T(0.5, \pm 1) = 6.5$.

At $x = \pm 1$, the extremum is at $y = 0$. $T(1, 0) = 10$ and $T(-1, 0) = 14$.

Now the boundaries of the boundaries. That's just $T(1, \pm 1) = 7$ and $T(-1, \pm 1) = 11$.

Phew. Now look at the list of temperatures and pick out the largest and smallest. The largest is $T(-1, 0) = 14$; the smallest is $T(0.5, \pm 1) = 6.5$.

11.3 Do the derivatives;

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

Now taking the v derivative,

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 1$$

Integrating this over u and v gives

$$w = uv + f_1(u) + f_2(v)$$

where f_1 and f_2 are arbitrary functions of u and v *only*.

11.10 If $f = u - sT$, then $df = T\,ds - p\,dv - s\,dT - T\,ds = -p\,dv - s\,dT$. Similarly, if $h = u + pv$, then $dh = T\,ds - p\,dv + p\,dv + v\,dp = T\,ds + v\,dp$. Finally, if $g = u - sT + pv$, both original differential terms will be eliminated.

12.13 This is straightforward.

$$\frac{d}{dx}\int_{1/x}^{2/x} dt\,\frac{\sin xt}{t} = \frac{\sin 1 - \sin 2}{x} + \left.\frac{\sin xt}{x}\right|_{1/x}^{2/x} = 0$$