Homework Solutions, Chapter 6

6.13 Part (a) asks for the temperature gradient, with ϕ the temperature. Find

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j}$$

At $(1, -\pi/4)$, this is

$$\nabla\phi(1, -\pi/4) = \frac{e}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

(b) We need the "directional derivative,"

$$\frac{d\phi}{ds} = \nabla\phi \cdot \frac{(\mathbf{i} + \mathbf{j}\sqrt{3})}{|\mathbf{i} + \mathbf{j}\sqrt{3}|}$$

At $(0, \pi/3)$, this is

$$\frac{d\phi}{ds}(0, \pi/3) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

(c) The electric field is $\mathbf{E} = -\nabla\phi$. At $(0, \pi)$, this is

$$\mathbf{E}(0, \pi) = \mathbf{i}$$

(d) At $(-1, y)$,

$$E(-1, y) = |\nabla\phi(-1, y)| = e^{-1}\sqrt{\cos^2 y + \sin^2 y} = e^{-1}$$

8.12 For conservative $\mathbf{F}$, $\nabla \times \mathbf{F} = 0$. Check:

$$\nabla \times (y\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}) = \mathbf{i}(0 - 0) + \mathbf{j}(0 - 0) + \mathbf{k}(1 - 1) = 0$$

For the potential, integrate to find the work, and insert a minus sign because $\mathbf{F} = -\nabla\phi$. Since for a conservative force, the work integral will be path-independent, pick an easy path, such as: go along the x axis from $(0, 0, 0)$ to $(x, 0, 0)$, then go parallel to the y axis to $(x, y, 0)$, and then parallel to z Along the first segment, $d\mathbf{r} = dx\,\mathbf{i}$, along the second, $d\mathbf{r} = dy\,\mathbf{j}$, and along the third, $d\mathbf{r} = dz\,\mathbf{k}$.

$$\phi = -\int_0^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \int_0^x dx\,0 + \int_0^y dy\,x + \int_0^z dz\,1 = -xy - z$$

We can verify that

$$\nabla(xy + z) = y\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k} = \mathbf{F}$$

10.3 We need to find

$$\iint \mathbf{r}\cdot\mathbf{n}\,d\sigma = \oint \mathbf{r}\cdot d\mathbf{A} = \int dv\,\nabla\cdot\mathbf{r}$$

With $\nabla\cdot\mathbf{r} = 3$, this becomes three times the volume of the cylinder: $3(\pi)3 = 9\pi$.

11.2 Do the curl straightforwardly for (a):

$$\nabla\times[(x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}] = \mathbf{i}(0-0) + \mathbf{j}(0-0) + \mathbf{k}(2y + 2y) = 4y\mathbf{k}$$

(b) Do the integral directly, using $d\sigma = \mathbf{k}\,dx\,dy$:

$$\iint(\nabla\times\mathbf{A})\cdot d\sigma = 4\int_0^a dx\int_0^b dy\,y = 4\int_0^a dx\,\frac{b^2}{2} = 2ab^2$$

(c) Do the line integral directly:

$$\begin{aligned}\oint \mathbf{A}\cdot d\mathbf{r} &= \int_0^a dx\,(x^2 - 0^2) + \int_0^b dy\,2ay + \int_a^0 dx\,(x^2 - b^2) + \int_b^0 dy\,0 \\ &= \frac{a^3}{3} + \frac{2ab^2}{2} - \frac{a^3}{3} + ab^2 = 2ab^2\end{aligned}$$