
Homework Solutions #4 (McIntyre Chapter 5)

2 This is straightforward.

(a) $\langle \psi | \psi \rangle = A^* A (1 + 1 + 1) = 1$. Choose $A = \frac{1}{\sqrt{3}}$.

(b) E_1, E_2 , and E_3 are possible, with equal probabilities of $\frac{1}{3}$.

(c) $\langle E \rangle = \frac{1}{3}(E_1 + E_2 + E_3) = \frac{14}{3}E_1$

(d) $|\psi(t)\rangle = A(e^{-i\omega_1 t}|\varphi_1\rangle - e^{-4i\omega_1 t}|\varphi_2\rangle + ie^{-9i\omega_1 t}|\varphi_3\rangle)$, where $\omega_1 = E_1/\hbar$.

(e) The same as (b), since H does not depend on t .

5 Obviously $\langle x \rangle = \frac{L}{2}$, but let's calculate everything.

$$\langle x \rangle = \langle \varphi_n | x | \varphi_n \rangle = \frac{2}{L} \int_0^L dx x \sin^2 \left(\frac{n\pi}{L} x \right) = \frac{L}{2}$$

$$\langle x^2 \rangle = \langle \varphi_n | x^2 | \varphi_n \rangle = \frac{2}{L} \int_0^L dx x^2 \sin^2 \left(\frac{n\pi}{L} x \right) = \left(\frac{1}{3} - \frac{1}{2n^2\pi^2} \right) L^2$$

$$\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2 \right)^{\frac{1}{2}} = \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} L$$

$$\langle p \rangle = \langle \varphi_n | p | \varphi_n \rangle = -i\hbar \frac{2}{L} \int_0^L dx \sin \left(\frac{n\pi}{L} x \right) \frac{\partial}{\partial x} \sin \left(\frac{n\pi}{L} x \right) =$$

$$-i\hbar \frac{2n\pi}{L^2} \int_0^L dx \sin \left(\frac{n\pi}{L} x \right) \cos \left(\frac{n\pi}{L} x \right) = 0$$

$$\langle p^2 \rangle = 2m\langle E \rangle = 2m\langle \varphi_n | H | \varphi_n \rangle = 2mE_n = \frac{n^2\pi^2\hbar^2}{L^2}$$

$$\Delta p = \left(\langle p^2 \rangle - \langle p \rangle^2 \right)^{\frac{1}{2}} = \frac{n\pi\hbar}{L}$$

8 Normalization means

$$\int_0^L dx \psi^* \psi = |A|^2 \int_0^L dx x^2 (L-x)^2 = 1 \quad \Rightarrow \quad |A| = \sqrt{\frac{30}{L^5}}$$

For the time evolution, we need to find the components of the state in the energy basis.

$$\langle \varphi_n | \psi(0) \rangle = \sqrt{\frac{2}{L}} \int_0^L dx \sin\left(\frac{n\pi}{L}x\right) Ax(L-x) = \frac{4\sqrt{15}}{(n\pi)^3} [1 - (-1)^n]$$

Since $\psi(x)$ is symmetric with respect to reflections about $x = \frac{L}{2}$, we would expect that all the even n components are zero, which is the case.

The time evolution is

$$\psi(x, t) = e^{-i\frac{t}{\hbar}\hat{H}}\psi(x, 0) = e^{-i\frac{t}{\hbar}\hat{H}} \sum_n \psi_n \varphi_n(x) = \sum_n \psi_n e^{-in^2\omega_1 t} \varphi_n(x)$$

where $\psi_n = \langle \varphi_n | \psi(0) \rangle$ and $\omega_1 = E_1/\hbar$. This doesn't look like it has any easier functional form.

The expectation value $\langle x \rangle = \frac{L}{2}$ always. Consider the operator \mathcal{Q} , which effects a reflection of a function about $x = \frac{L}{2}$. (Clearly \mathcal{Q} is closely related to parity.) $\psi(x, 0)$ is a \mathcal{Q} eigenstate, and $[H, \mathcal{Q}] = 0$. Therefore $\psi(x, t)$ is also a \mathcal{Q} eigenstate, and all \mathcal{Q} eigenstates give $\langle x \rangle = \frac{L}{2}$.

But that might not be obvious, so here's a brute force calculation:

$$\begin{aligned} \langle x \rangle &= \int_0^L dx \left(\sum_n \psi_n^* e^{in^2\omega_1 t} \varphi_n^* \right) x \left(\sum_m \psi_m e^{-im^2\omega_1 t} \varphi_m \right) = \\ &= \sum_{nm} \psi_n^* \psi_m e^{i(n^2-m^2)\omega_1 t} \int_0^L dx \varphi_n^*(x) x \varphi_m(x) \end{aligned}$$

If you do the integral,

$$\int_0^L dx \varphi_n^*(x) x \varphi_m(x) = \frac{2}{L} \int_0^L dx x \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) = \frac{L}{2} \delta_{nm}$$

Putting this in,

$$\langle x \rangle = \frac{L}{2} \sum_n \psi_n^* \psi_n = \frac{L}{2}$$

since the sum above is the sum of all energy probabilities, which is normalized to 1.

17 The probabilities for $x > a$ and $x < -a$ are the same. Therefore

$$\mathcal{P}_{|x|>a} = 2A^2 \int_a^\infty dx e^{-qx} = \frac{A^2 e^{-2qa}}{q} = \frac{aA^2 e^{-2\sqrt{z_0^2 - z^2}}}{\sqrt{z_0^2 - z^2}}$$

We now need to normalize the ground state to find A^2 .

$$\mathcal{P}_{|x|<a} = D^2 \int_{-a}^a dx \cos^2(kx) = aD^2 \left[\frac{\sin(2ka)}{2ka} + 1 \right] = aD^2 \left[\frac{\sin(2z)}{2z} + 1 \right]$$

Since the probabilities sum to 1,

$$\frac{aA^2 e^{-2\sqrt{z_0^2 - z^2}}}{\sqrt{z_0^2 - z^2}} + aD^2 \left[\frac{\sin(2z)}{2z} + 1 \right] = 1$$

From the quantization conditions for even parity (the ground state is even), we also get, $Ae^{-qa} = D \cos ka$ and $qAe^{-qa} = kD \sin ka$. Using these, we get, after some algebra not worth reproducing,

$$\mathcal{P}_{|x|>a} = \frac{\cos^2 z}{1 + \sqrt{z_0^2 - z^2}}$$