Homework Solutions #5 (McIntyre Chapter 6)

2ii Just like the recitation question:

(a) Normalization:

$$|A|^2 \int dx\, x^2 e^{-2x^2/a^2} = 1 \qquad \Rightarrow \qquad A = \pi^{-\frac{1}{4}} 2^{\frac{5}{4}} a^{-\frac{3}{2}}$$

(b) Since the integrand is an odd function,

$$\langle x \rangle = |A|^2 \int dx\, x^3 e^{-2x^2/a^2} = 0$$

(c)

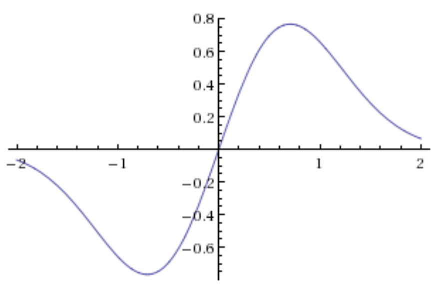
$$\langle x^2 \rangle = |A|^2 \int dx\, x^4 e^{-2x^2/a^2} = |A|^2 3\pi^{\frac{1}{2}} 2^{-\frac{9}{2}} a^5$$

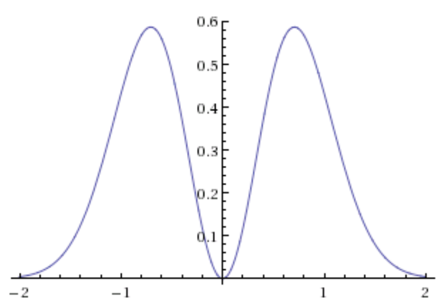
$$\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2} a$$

(d)

$$\mathcal{P}_{0<x<a} = |A|^2 \int_0^a dx\, x^2 e^{-2x^2/a^2} = 0.369$$

(e) Plots with x/a on the horizontal axis:





(f) Since ψ is an odd function, and a derivative inverts the parity, the integral will give zero:

$$\langle p \rangle = |A|^2 \int dx\, \left(x e^{-x^2/a^2}\right) \left(-i\hbar \frac{\partial}{\partial x}\right) \left(x e^{-x^2/a^2}\right) = 0$$

(g)

$$\langle p^2 \rangle = |A|^2 \int dx \, \left(xe^{-x^2/a^2}\right) \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \left(xe^{-x^2/a^2}\right) =$$

$$-\hbar^2 |A|^2 \int dx \, e^{-2x^2/a^2} \left(-6\frac{x^2}{a^2} + 4\frac{x^4}{a^4}\right) = \frac{3\hbar^2}{a^2}$$

$$\Delta p = (\langle p^2 \rangle - \langle p \rangle^2)^{\frac{1}{2}} = \frac{\sqrt{3}\hbar}{a}$$

(h)

$$\Delta x \Delta p = \frac{3}{2}\hbar \geq \frac{\hbar}{2}$$

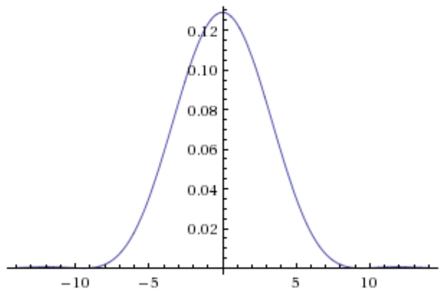
4 Fourier transform:

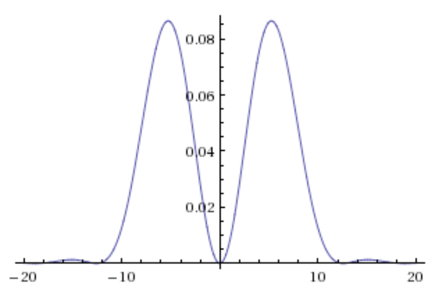
$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_0^L dx \, e^{-ikx} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) = \frac{1}{2i\sqrt{\pi L}} \int_0^L dx \, \left[e^{i(\frac{n\pi}{L} - k)x} - e^{-i(\frac{n\pi}{L} + k)x}\right]$$

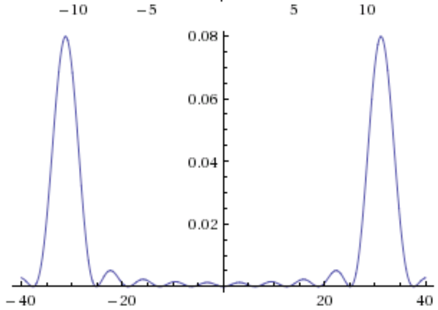
$$= \sqrt{\frac{L}{\pi}} \left\{ \frac{n\pi}{n^2\pi^2 - k^2L^2} \left[1 - (-1)^n e^{-ikL}\right] \right\}$$

$$P(k) = |\phi(k)|^2 = \frac{n^2 2\pi L \left[1 - (-1)^n \cos kL\right]}{\left[n^2\pi^2 - k^2L^2\right]^2}$$

Plots with kL on the horizontal axis:







Note that the probabilities peak at $kL = \pm n\pi$, where the overlap is largest. In the $n = 1$ case, the $\pm\pi$ peaks are close together, so the peaks merge into one.

26 Resonance means

$$\frac{2a}{\hbar}\sqrt{2m_e(E+V_0)} = n\pi$$

Therefore

$$\frac{a}{n} = \frac{\pi\hbar c}{2\sqrt{2m_ec^2(E+V_0)}} = \frac{\pi \cdot 197.3 \text{ eV} \cdot \text{nm}}{\left(2\sqrt{2(0.511)19}\right) 10^3 \text{ eV}} = 0.0703 \text{ nm}$$

5 bound states means $2\pi < z_0 < \frac{5}{2}\pi$, so

$$2\pi < \frac{\sqrt{2m_eV_0}}{\hbar}a < \frac{5}{2}\pi \qquad \Rightarrow \qquad 0.434 \text{ nm} < a < 0.542 \text{ nm}$$

This means $n = 7$ and $a = 0.493$ nm.

For other resonances, use

$$E = \frac{n^2\pi^2(\hbar c)^2}{8a^2(m_ec^2)} - V_0 = n^2(0.388 \text{ eV}) - 8 \text{ eV}$$

The minimum $n = 5$, for which $E = 1.7$ eV, and so forth.

29 First, for $E < V_0$: For $x < 0$, $\phi_I = Ae^{ikx} + Be^{-ikx}$ where $\frac{\hbar^2k^2}{2m} = E$. For $x > 0$, $\phi_{II} = Ce^{-qx}$ where $\frac{\hbar^2q^2}{2m} = V_0 - E$. The boundary conditions at $x = 0$ are

$$A + B = C \qquad ikA - ikB = -qC$$

Dividing by A,

$$1 + \frac{B}{A} = \frac{C}{A} \qquad 1 - \frac{B}{A} = i\frac{q}{k}\frac{C}{A}$$

Solving, we get

$$\frac{C}{A} = \frac{2}{1+i\frac{q}{k}} \qquad \frac{B}{A} = \frac{1-i\frac{q}{k}}{1+i\frac{q}{k}}$$

$$R = \left|\frac{B}{A}\right|^2 = 1$$

Now for $E > V_0$. For $x < 0$, $\phi_I = Ae^{ik_1 x} + Be^{-ik_1 x}$ where $\frac{\hbar^2 k_1^2}{2m} = E$. For $x > 0$, $\phi_{II} = Ce^{ik_2 x}$ where $\frac{\hbar^2 k_2^2}{2m} = E - V_0$. The boundary conditions at $x = 0$ are

$$A + B = C \qquad ik_1 A - ik_1 B = ik_2 C$$

Dividing by A,

$$1 + \frac{B}{A} = \frac{C}{A} \qquad 1 - \frac{B}{A} = \frac{k_2}{k_1}\frac{C}{A}$$

Solving, we get

$$\frac{C}{A} = \frac{2}{1 + \frac{k_2}{k_1}} \qquad \frac{B}{A} = \frac{1 - \frac{k_2}{k_1}}{1 + \frac{k_2}{k_1}}$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{\left(\sqrt{E} - \sqrt{E - V_0}\right)^2}{\left(\sqrt{E} + \sqrt{E - V_0}\right)^2}$$

Plotted with E/V_0 on the horizontal axis, R looks like