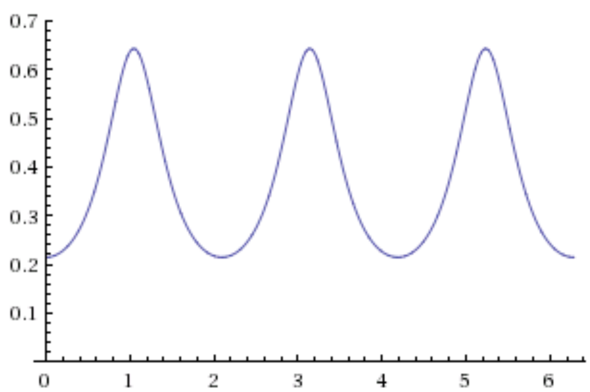
Homework Solutions #6 (McIntyre Ch 7 & 11)

7.15

(a) You can do the normalization integral as a contour integral, but it's a pretty standard form that is easily doable by common software such as Mathematica. You get:

$$\int d\phi\, \psi^* \psi = \int_0^{2\pi} d\phi\, \frac{|N|^2}{(2+\cos(3\phi))^2} = \frac{4\pi |N|^2}{3\sqrt{3}} = 1 \quad \Rightarrow \quad N = \frac{3^{\frac{3}{4}}}{2\sqrt{\pi}}$$



(b)

(c)

$$\langle L_z \rangle = -i\hbar \int_0^{2\pi} d\phi\, \frac{N^*}{2+\cos(3\phi)} \frac{\partial}{\partial\phi} \left(\frac{N}{2+\cos(3\phi)} \right) = 0$$

You can see the integral has to be 0 because $\psi(\phi)$ is real, therefore the integral is imaginary, and $\langle L_z \rangle$ has to be real.

7.26 Using $e^{i\phi} = \cos\phi + i\sin\phi$, it's easy to verify the given form for $L_\pm$. Now, using these operators,

$$L_\pm Y_1^0 = \hbar e^{\pm i\phi} \sqrt{\frac{3}{4\pi}} (\mp \sin\theta) = \hbar\sqrt{2}\, Y_1^{\pm 1}$$

$$L_\pm Y_1^{\pm 1} = 0$$

$$L_\pm Y_1^{\mp 1} = \hbar\sqrt{\frac{3}{8\pi}}(2\cos\theta) = \hbar\sqrt{2}\,Y_1^0$$

All of these fit the pattern in

$$L_\pm Y_1^m = \hbar\sqrt{1(1+1) - m(m\pm 1)}\,Y_1^{m\pm 1} \quad \text{or}$$

$$L_\pm|1m\rangle = \hbar\sqrt{1(1+1) - m(m\pm 1)}\,|1, m\pm 1\rangle$$

7.29 Most of these probabilities can be read directly off the coefficients of the given state.

(a) There is no component of $|\psi\rangle$ with $m = 2$, but there are $m = -1$ and $m = 0$ components.

$$\mathcal{P}_{L_z=2\hbar} = 0 \qquad \mathcal{P}_{L_z=-\hbar} = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \qquad \mathcal{P}_{L_z=0\hbar} = \left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{i}{\sqrt{6}}\right|^2 = \frac{1}{2}$$

(b) Since we already have the probabilities,

$$\langle L_z\rangle = \mathcal{P}_{L_z=-\hbar}(-\hbar) + \mathcal{P}_{L_z=0\hbar}(0\hbar) = -\frac{\hbar}{2}$$

(c) $l = 1$ and $l = 0$ is possible, with probabilities easily read off from $|\psi\rangle$ to be $\frac{5}{6}$ and $\frac{1}{6}$. Therefore

$$\langle \mathbf{L}^2\rangle = \mathcal{P}_{l=1}(\hbar^2 1(1+1)) + \mathcal{P}_{l=0}(\hbar^2 0(0+1)) = \frac{5\hbar^2}{3}$$

(d) $\langle E\rangle = \langle \frac{1}{2I}\mathbf{L}^2\rangle = \frac{1}{2I}\langle \mathbf{L}^2\rangle = \frac{5\hbar^2}{6\mu r_0^2}$

(e) Using $L_y = \frac{1}{2i}(L_+ - L_-)$, we get

$$\langle L_y\rangle = \left(\frac{1}{\sqrt{2}}\langle 1,-1| + \frac{1}{\sqrt{3}}\langle 10| - \frac{i}{\sqrt{6}}\langle 00|\right)\frac{1}{2i}(L_+ - L_-)$$

$$\left(\frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{i}{\sqrt{6}}|00\rangle\right) = 0$$

11.4

(a) Eigenstates: $|m\rangle$, with $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$. Eigenvalue equations:

$$\mathbf{S}^2|m\rangle = \frac{15}{4}\hbar^2|m\rangle \qquad S_z|m\rangle = m\hbar|m\rangle$$

(b)

$$\mathbf{S}^2 \doteq \frac{15}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(c) Calculating the matrix elements $\langle m|S_+|n\rangle$, we get

$$S_+ \doteq \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad S_- = S_+^\dagger \doteq \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Therefore

$$S_x = \frac{1}{2}(S_+ + S_-) \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}$$

(d) The eigenvalues should be $m\hbar$, with $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

$$\begin{vmatrix} -\lambda & \sqrt{3} & 0 & 0 \\ \sqrt{3} & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -\lambda \end{vmatrix} = \lambda^2(\lambda^2-3)-4\lambda^2-3(\lambda^2-3) = (\lambda^2-9)(\lambda^2-1) = 0$$

The solutions, including the factor of $\frac{\hbar}{2}$, are $m\hbar$, with $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.