Solutions to Second Midterm

1. (60 points) Consider a particle confined to a one-dimensional "box" of length L. The quantum state of the particle is described by a complex wave function $\psi(x)$, where $\psi(x) = 0$ for $x \leq 0$ and $x \geq L$ (outside the box). Every possible wave function can be expanded in terms of a set of basis functions $\varphi_n(x)$, $n = 1, 2, 3, \ldots$, each which corresponds to a different energy level.

We express all this by defining an inner product between states ψ and ξ:

$$\langle \psi | \xi \rangle = \int_0^L dx\, \psi^*(x) \xi(x)$$

And then every state $\psi(x)$ can be expanded as

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \varphi_n(x)$$

where the components ψ_n are complex numbers, and the orthonormal basis functions are

$$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \quad \text{and} \quad \langle \varphi_n | \varphi_m \rangle = \int_0^L dx\, \varphi_n^*(x) \varphi_m(x) = \delta_{nm}$$

Now say you have a particle in a state

$$\psi(x) = \begin{cases} \sqrt{2/L} & \frac{L}{4} < x < \frac{3L}{4} \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the components ψ_n, $n = 1, 2, 3, \ldots$, for this state. *Note:* This *is* a Fourier series problem! It just appears in a slightly different context which I hope will force you to think about what an expansion in terms of orthogonal functions means.

Answer: Using the orthogonality relationship,

$$\int_0^L dx\, \varphi_m^*(x) \left(\sum_{n=1}^{\infty} \psi_n \varphi_n(x) \right) = \sum_{n=1}^{\infty} \psi_n \left(\int_0^L dx\, \varphi_m^*(x) \varphi_n(x) \right)$$

$$= \sum_{n=1}^{\infty} \psi_n \delta_{mn} = \psi_m$$

Therefore the components are obtained by the following Fourier integral:

$$\psi_n = \int_0^L dx\, \varphi_n^*(x)\psi(x)$$

In our case,

$$\begin{aligned} \psi_n &= \frac{2}{L}\int_{\frac{L}{4}}^{\frac{3L}{4}} dx\, \sin\left(\frac{n\pi}{L}x\right) = -\frac{2}{L}\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_{\frac{L}{4}}^{\frac{3L}{4}} \\ &= \frac{2}{n\pi}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right) \end{aligned}$$

(b) The magnitude squares of the components, $|\psi_n|^2$, are the probabilities of measuring the energy of a particle to be E_n. Find $P(E_1) = |\psi_1|^2$. (Calculate the actual number.)

Answer: With $n = 1$,

$$|\psi_1|^2 = \frac{4}{\pi^2}\left(\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}\right)^2 = \frac{8}{\pi^2} = 0.81$$

(c) $\phi(k)$, which is the momentum-space representation of the state $\psi(x)$, is the Fourier transform

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\psi(x)$$

The probability distribution $P(k) = |\phi(k)|^2 = \phi^*(k)\phi(k)$ gives the probability of measuring the momentum to be $\hbar k$. Find $P(k)$. Make sure no i's appear in your answer!

Answer: Since $\psi(x) = 0$ everywhere except $\frac{L}{4} < x < \frac{3L}{4}$,

$$\begin{aligned} \phi(k) &= \frac{1}{\sqrt{2\pi}}\frac{2}{L}\int_{\frac{L}{4}}^{\frac{3L}{4}} dx\, e^{-ikx} = \frac{1}{\sqrt{\pi L}}\frac{e^{-ikx}}{-ik}\Big|_{\frac{L}{4}}^{\frac{3L}{4}} \\ &= \frac{i}{k\sqrt{\pi L}}\left(e^{-i\frac{3}{4}kL} - e^{-i\frac{1}{4}kL}\right) \end{aligned}$$

Therefore

$$
\begin{aligned}
P(k) &= \phi^*(k)\phi(k) = \frac{-i^2}{\pi L k^2}\left(e^{-i\frac{3}{4}kL} - e^{-i\frac{1}{4}kL}\right)\left(e^{i\frac{3}{4}kL} - e^{i\frac{1}{4}kL}\right) \\
&= \frac{2}{\pi L}\frac{1}{k^2}\left(1 - \cos\frac{kL}{2}\right)
\end{aligned}
$$

2. (40 points) You have an RLC circuit to which you apply an exponentially decaying voltage $V = V_0 e^{-\gamma t}$. The differential equation describing the current in this circuit is

$$\frac{d^2 I}{dt^2} + a\frac{dI}{dt} + bI = -f\, e^{-\gamma t}$$

where $a = R/L$, $b = 1/LC$, and $f = V_0\gamma/L$. Let's say your circuit elements have values such that your differential equation ends up being

$$I'' + 2I' + I = -e^{-2t}$$

You also have the initial conditions that at $t = 0$, $I(0) = 0$ and $I'(0) = 0$. Find $I(t)$.

Answer: Since $a^2 - 4b = 4 - 4 = 0$, the solutions to $\frac{1}{2}(a \pm \sqrt{a^2 - 4b}) = 1$ coincide. The homogenous solution must look like

$$I_h = A\, e^{-t} + B\, t\, e^{-t}$$

For a particular solution, try a solution that looks like $I_p = g\, e^{-\gamma t} = g\, e^{-2t}$. Putting that into the differential equation, we get

$$4g\, e^{-2t} - 4g\, e^{-2t} + g\, e^{-2t} = -e^{-2t} \qquad \Rightarrow \qquad g = -1$$

So the solution is

$$I(t) = I_p + I_h = -e^{-2t} + A\, e^{-t} + B\, t\, e^{-t}$$

To determine A and B, we use the initial conditions. First, we need I'.

$$I'(t) = 2\, e^{-2t} - A\, e^{-t} - B\, t\, e^{-t} + B\, e^{-t}$$

Now, at $t = 0$,

$$I(0) = -1 + A = 0 \qquad \Rightarrow \qquad A = 1$$

$$I'(0) = 1 + B = 0 \qquad \Rightarrow \qquad B = -1$$

Therefore

$$I(t) = -e^{-2t} + e^{-t} - t\,e^{-t}$$