Homework Solutions, Chapter 7

5.5 Doing the integrals for the Fourier components:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[-\int_0^{\frac{\pi}{2}} dx\ \cos nx + \int_{\frac{\pi}{2}}^{\pi} dx\ \cos nx\right] \\ &= \frac{1}{n\pi}\left[\sin 0 - \sin(n\pi/2) + \sin(n\pi) - \sin(n\pi/2)\right] \end{aligned}$$

Since $\sin(n\pi) = 0$ and $\sin(n\pi/2) = +1, 0, -1, 0, +1, 0\ldots$ for $n = 1, 2, 3, 4, 5, 6, \ldots$, the cosine series goes like

$$-\frac{2}{\pi}\cos x + \frac{2}{3\pi}\cos 3x - \frac{2}{5\pi}\cos 5x - \ldots$$

The sine part:

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[-\int_0^{\frac{\pi}{2}} dx\ \sin nx + \int_{\frac{\pi}{2}}^{\pi} dx\ \sin nx\right] \\ &= \frac{1}{n\pi}\left[\cos(n\pi/2) - \cos 0 - \cos(n\pi) + \cos(n\pi/2)\right] \end{aligned}$$

So the b_n's are:

$$b_n = \frac{1}{n\pi}\left[-1 - (-1)^n + 2\cos(n\pi/2)\right] = 0, -\frac{2}{\pi}, 0, 0, 0, -\frac{4}{6\pi}\ldots$$

The sine part of the Fourier series is then

$$-\frac{2}{\pi}\sin 2x - \frac{4}{6\pi}\sin 6x + \ldots$$

7.12 Since $f(x)$ is real, $f(x) = f^*(x)$. So

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n^* e^{-inx}$$

Now notice that the sum includes *all* positive and negative n, so we can do the sum backwards, replacing all n with $-n$:

$$\sum_{n=-\infty}^{\infty} c_n^* e^{-inx} = \sum_{n=-\infty}^{\infty} c_{-n}^* e^{inx}$$

Comparing this to the first equation, and equating the coefficients of e^{inx} on each side, we get $c_n = c_{-n}^*$.

8.13 First, $f(x) = 2-x, -2 < x < 2$. It's easiest to start with the complex Fourier coefficients:

$$c_n = \frac{1}{4}\int_{-2}^{2} dx\,(2-x)\,e^{-in\pi x/2} = -\frac{i}{2n\pi}\,e^{-in\pi x/2}\left(x+\frac{2}{in\pi}\right)_{-2}^{2} = \frac{2i(-1)^{n+1}}{n\pi}$$

$n = 0$ is an exception, with

$$c_0 = \frac{1}{2}a_0 = \frac{1}{4}\int_{-2}^{2} dx\,(2-x) = 2$$

Now, we can get, for $n > 0$,

$$a_n = c_n + c_{-n} = 0$$

$$b_n = i(c_n - c_{-n}) = \frac{4(-1)^n}{n\pi}$$

Then, $f(x) = 2 - x, 0 < x < 4$.

$$c_n = \frac{1}{4}\int_{0}^{4} dx\,(2-x)\,e^{(1-in)x} = -\frac{i}{2n\pi}\,e^{-in\pi x/2}\left(x+\frac{2}{in\pi}\right)_{0}^{4} = -\frac{2i}{n\pi}$$

$\frac{1}{2}a_0 = c_0 = 0$ now, and

$$a_n = c_n + c_{-n} = 0$$

$$b_n = i(c_n - c_{-n}) = \frac{4}{n\pi}$$

9.1

$$e^{inx} = \frac{1}{2}(e^{inx} + e^{-inx}) + \frac{1}{2}(e^{inx} - e^{-inx}) = \sin nx + i\cos nx$$

$$xe^{x} = \frac{1}{2}(xe^{x} - xe^{-x}) + \frac{1}{2}(xe^{x} + xe^{-x}) = x\sinh x + x\cosh x$$

12.6 The Fourier transform is

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} dx\, e^{-ikx} x \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{i}{k}\left(e^{-ik}+e^{ik}\right) - \left(\frac{i}{k}\right)^2\left(e^{-ik}-e^{ik}\right)\right] \\ &= \frac{2i}{\sqrt{2\pi}}\left(\frac{1}{k}\cos k - \frac{1}{k^2}\sin k\right)\end{aligned}$$

The Fourier integral is then

$$f(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\left(\frac{1}{k}\cos k - \frac{1}{k^2}\sin k\right)$$

If you use the book's non-symmetric transform,

$$g(\alpha) = \frac{i}{\pi}\left(\frac{1}{\alpha}\cos\alpha - \frac{1}{\alpha^2}\sin\alpha\right)$$

Note that this is purely imaginary, as it should be for a transform of an odd function.

13.12 The largest Fourier component in the series is $n = 3$, since then the denominator, $100(n-3)^2 + 1 = 1$, while it is larger than 100 for any other n. Nothing else comes within a percentage of $n = 3$. This component has angular frequency $\omega = 180\pi$, so $f = 90$ Hz (if t was in seconds) is the apparent frequency.