Homework Solutions, Chapter 8

1.4 Integrate d^2x/dt^2 twice to get $dx/dt = -\frac{g}{k}e^{-kt}+c_1$ and $x = \frac{g}{k^2}e^{-kt}+c_1t+c_2$, where c_1 and c_2 are constants to be determined by the initial conditions. Since it starts from rest, $dx/dt = 0$ when $t = 0$. This gives

$$-\frac{g}{k} + c_1 = 0 \Rightarrow c_1 = \frac{g}{k}$$

Say that $x(0) = 0$, so

$$\frac{g}{k^2} + c_2 = 0 \Rightarrow c_2 = -\frac{g}{k^2}$$

For small t, use the Maclaurin series for e^{-kt}, resulting in

$$x = \frac{g}{k^2}\left(1 - kt + \frac{(kt)^2}{2} - \dots\right) + \frac{gt}{k} - \frac{g}{k^2}$$

Notice that c_1t and c_2 cancel the first two terms in the expansion, leaving

$$x = \frac{1}{2}gt^2 + \dots$$

As $t \to \infty$, $\frac{dx}{dt} \to g/k$.

2.16 Since $\frac{dy}{dx} = y^{\frac{1}{2}}$, $\int dy\, y^{-\frac{1}{2}} = \int dx$, and so

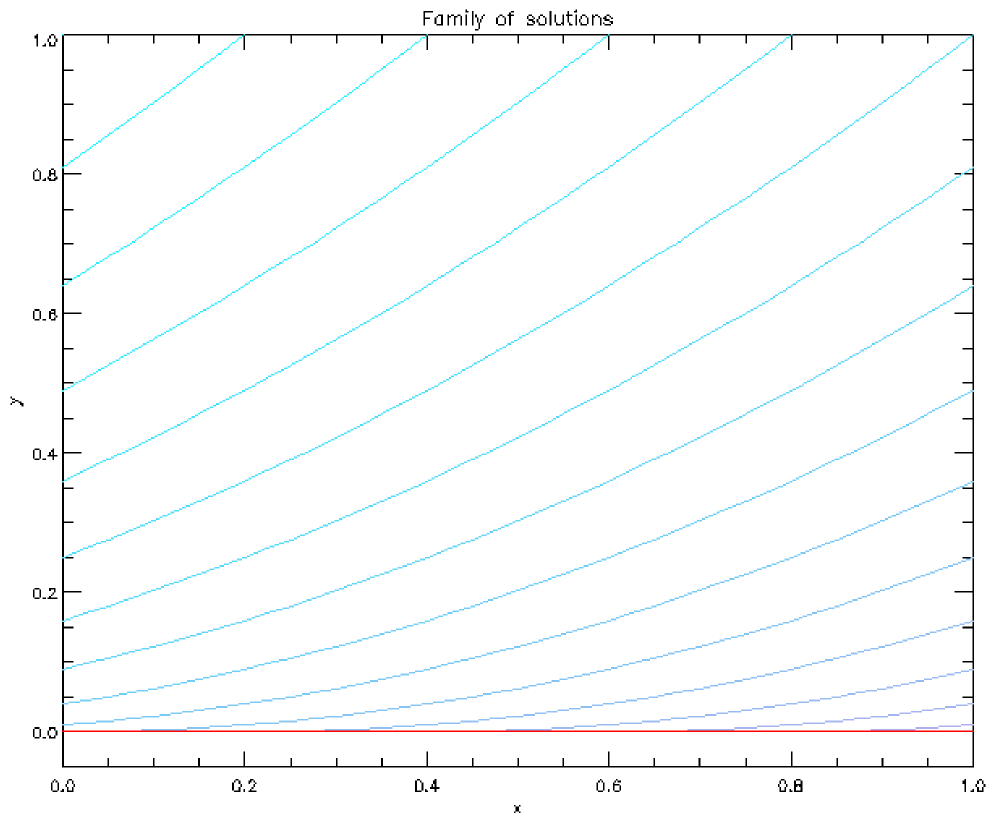
$$y = \left(\frac{x}{2} - c\right)^2$$

with c an arbitrary constant. A particular solution at $x = 0$ is either the solution with $c = 0$, or the solution $y = 0$ for all x. Naturally, $y = 0$ is not obtained by setting c to any value. So the solutions look like a family of parabolas all with their minima on the x axis, and $y = 0$ tangent to the lot.

Now, since $\sqrt{y} \geq 0$, the slope y' must always be non-negative. So only the positive slope parts of the parabolas are solutions—their right halves. For any point with $y > 0$, only one solution curve passes through that point. But on the x axis, with $y = 0$ and $x = k$, any solution

$$y = \begin{cases} 0 & x < 2c \\ \left(\frac{x}{2} - c\right)^2 & x \geq 2c \end{cases}$$

will work, provided that $2c \geq k$. These are infinite in number.



3.2 Following the book's procedure in equation (3.9) will produce a solution, after dividing everything by x^2. Then $P = 3/x$ and $Q = 1/x^2$; $I = \int dx\, P = 3 \ln x$, so $e^I = x^3$. Then

$$y = \frac{1}{x^3}\left(\int dx\, x + c\right) = \frac{1}{2x} + \frac{c}{x^3}$$

5.8 $Dy = 0$ is solved by any constant A. $(D + 5)y = 0$ is solved by $y = Be^{-5x}$. So the general solution is

$$y = A + Be^{-5x}$$

5.35 Simple harmonic motion occurs with a force $F = -kx$. Using $F =$

$m\, d^2x/dt^2$, we get the ODE

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution to this is $x = A\sin(\sqrt{k/m}t + \phi)$—periodic motion with period $T = 2\pi\sqrt{m/k}$. Writing the differential equation for this problem, we see mg/R is analogous to k, producing

$$\frac{d^2r}{dt^2} + \frac{g}{R}r = 0$$

This gives simple harmonic motion with period $T = 2\pi\sqrt{R/g}$.

6.16 First solve the homogeneous equation, $(D-3i)(D+3i)y = 0$. This is solved by $y_c = Ae^{3ix} + Be^{-3ix}$. Then look for a particular solution to $(D-3i)(D+3i)y = 30e^{3ix}$. Notice that since e^{3ix} already satisfies the homogeneous equation, y_p cannot have this form. Try $y_p = cxe^{3ix}$ instead. Putting this into the differential equation, we get

$$(3ic + 3ic)e^{3ix} - 9cxe^{3ix} + 9cxe^{3ix} = 30e^{3ix}$$

which means $c = -5i$. The general solution is the imaginary part of $y = -5ixe^{3ix} + Ae^{3ix} + Be^{-3ix}$, which is

$$y = -5x\cos 3x + E\sin(3x + \phi)$$

with E and ϕ arbitrary constants.

7.5 Substituting $u = dy/dx$, we get $u' = k\sqrt{1+u^2}$. Separation of variables gives

$$\int \frac{du}{\sqrt{(1+u^2)}} = k\int dx$$

Doing the integrals gives

$$\sinh^{-1} u = kx + c_1 \Rightarrow u = \sinh(kx + c_1)$$

Integrating again,

$$y = \int dx\, \sinh(kx + c_1) = \frac{1}{k}\cosh(kx + c_1) + c_2$$

13.34 If the mass of the chain on one side of the string is m_1 and the other m_2, then the difference in weight is $(m_1 - m_2)g$. This is equal to $(m_1 + m_2)a$. Since the chain has a uniform linear density ρ, the differential equation is

$$2y\rho g = l\rho\frac{d^2y}{dt^2}$$

Cancelling the ρ's, the differential equation is

$$(D + \sqrt{2g/l})(D - \sqrt{2g/l})y = 0$$

Which has a solution

$$y = Ae^{\sqrt{2g/l}\,t} + Be^{-\sqrt{2g/l}\,t}$$

The initial conditions are that at $t = 0$, $dy/dx = 0$ and $y = y_0$. This means

$$A\sqrt{2g/l}\,e^0 - B\sqrt{2g/l}\,e^0 = 0 \Rightarrow A = B$$

$$Ae^0 + Be^0 = y_0 \Rightarrow A = B = \frac{y_0}{2}$$

Putting it all together:

$$y = y_0\,\frac{1}{2}\left(e^{\sqrt{2g/l}\,t} + e^{-\sqrt{2g/l}\,t}\right) = y_0 \cosh\sqrt{2g/l}\,t$$