1. (40 points) Derive equation 4.10 in your book, for the efficiency of an Otto cycle using an ideal gas as the working fluid. (Note: this derivation is easily available online. I trust you to do this yourself, consulting nothing but your book and notes.)

Answer: Q_h is taken in during the ignition part of the cycle, at constant V and therefore $W = 0$. Using the labeling in Figure 4.5 in the book,

$$Q_h = U_3 - U_2 = \frac{f}{2}Nk(T_3 - T_2) = \frac{f}{2}V_2(P_3 - P_2)$$

Similarly, the exhaust heat is

$$Q_c = U_4 - U_1 = \frac{f}{2}Nk(T_4 - T_1) = \frac{f}{2}V_1(P_4 - P_1)$$

We need to get rid of the pressures, so we can use the fact that PV^γ is constant during the adiabatic parts of the cycle. So $P_4 = P_3(V_2/V_1)^\gamma$ and $P_1 = P_2(V_2/V_1)^\gamma$. Therefore

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{V_1}{V_2}\frac{P_3 - P_2}{P_3 - P_2}\left(\frac{V_2}{V_1}\right)^\gamma = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

2. (60 points) We discussed how negative temperatures are warmer than positive temperatures, by showing how a two-state paramagnet goes from positive to negative temperatures by *adding* energy. Now investigate this sense that negative temperatures are warmer in a more fundamental sense.

(a) Take two two-state paramagnet systems A and B, starting at temperatures $T_A > 0$ and $T_B < 0$. Make any other assumptions you need about system parameters. Then bring A and B into thermal contact, so that they can exchange energy. Find the equilibrium temperature of the combined system, and use your calculation to reach a conclusion about whether negative temperatures are warmer.

Answer: Individual answers will differ, but this is one way to do it.

Take one system to have N_A and U_A, with $U_A > 0$ and therefore initial temperature $T_A < 0$. The other will have N_B and U_B, with $U_B < 0$ and therefore $T_B < 0$. Now look at the temperatures.

$$\frac{1}{T_A} = \frac{k}{\mu B}\ln\left(\frac{N_A - U_A/\mu B}{N_A + U_A/\mu B}\right) = \frac{k}{\mu B}\ln\left[\frac{1 - U_A/(N_A\mu B)}{1 + U_A/(N_A\mu B)}\right]$$

Note that the combined system will have an equilibrium temperature for $N = N_A + N_B$ and $U = U_A + U_B$:

$$\frac{1}{T_{A+B}} = \frac{k}{\mu B} \ln \left[\frac{1 - (U_A + U_B)/[(N_A + N_B)\mu B]}{1 + (U_A + U_B)/[(N_A + N_B)\mu B]} \right]$$

Now, with out our initial assumptions, when we compare,

$$\frac{U_A}{N_A} > \frac{U_A + U_B}{N_A + N_B} \qquad \Rightarrow \qquad \frac{1}{T_A} < \frac{1}{T_{A+B}}$$

A very similar argument gives

$$\frac{U_B}{N_B} < \frac{U_A + U_B}{N_A + N_B} \qquad \Rightarrow \qquad \frac{1}{T_B} > \frac{1}{T_{A+B}}$$

Therefore, $\frac{1}{T_A}$ increases toward $\frac{1}{T_{A+B}}$ after thermal contact, just like a warmer object coming into contact with a cooler one. But this involves going from a negative initial temperature to even more negative temperatures, and possibly through $T = \infty$ before reaching declining positive temperatures.

Similarly, $\frac{1}{T_B}$ decreases toward $\frac{1}{T_{A+B}}$ like a cold object warming up. This means T_B increases, but it can possibly go through $T = \infty$ to get to even hotter negative temperatures.

(b) You will see that the rule that heat spontaneously flows from A to B if $T_A > T_B$ is not correct. Find a more correct expression for this rule, and explain why your rule is better by using the concept of entropy.

Answer: From (a), you can see that the behavior of $1/T$ is key. Therefore, heat flows spontaneously from A to B if

$$\frac{1}{T_A} < \frac{1}{T_B}$$

This accounts for negative temperatures as well.

The reason is that you're maximizing entropy and it's $1/T = \partial S/\partial U$.