1. (50 points) You have a spin-$\frac{1}{2}$ system in $|\psi\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}\, e^{i\phi}|-\rangle$.

(a) By repeated measurements of particles prepared in the same state $|\psi\rangle$, you determine that $\mathcal{P}_{S_x=\hbar/2} = p_x$, $\mathcal{P}_{S_y=\hbar/2} = p_y$, and $\mathcal{P}_{S_z=\hbar/2} = p_z$. Show that if you get the same set of probabilities p_x, p_y, p_z for another state $|\varphi\rangle$, then $|\varphi\rangle = e^{i\xi}|\psi\rangle$, where ξ is a real constant. In other words, show that knowing p_x, p_y, p_z determines θ and ϕ uniquely, except for a physically inconsequential overall phase for the state.

Remember: Square roots and inverse trigonometric functions are multiple-valued functions!

Answer: The probabilities are

$$p_x = |{}_x\langle +|\psi\rangle|^2 = \tfrac{1}{2}\,(1 + \sin\theta\cos\phi)$$

$$p_y = |{}_y\langle +|\psi\rangle|^2 = \tfrac{1}{2}\,(1 + \sin\theta\sin\phi)$$

$$p_z = |\langle +|\psi\rangle|^2 = \tfrac{1}{2}\,(1 + \cos\theta)$$

We can take $0 \le \theta \le \frac{\pi}{2}$, such that $\sin\theta \ge 0$ and $\cos\theta \ge 0$. This is because if we take states with other θ angles that give the same p_z, either the state differs from $|\psi\rangle$ in its overall phase, which is unimportant, or it involves a relative phase that can be absorbed into ϕ. Given that, to also get p_x and p_y the same, $\sin\phi$ and $\cos\phi$ must also be the same, which is only possible with the same $0 \le \phi < 2\pi$.

(b) You prepare particles in state $|\psi\rangle$ by measuring an observable O, where $O|\psi\rangle = a|\psi\rangle$, and where the eigenvalues of O are a, b, with $a \ne b$. You then use a particle only after a measurement resulting in $O = a$. Find the matrix representation of O in the $\{|+\rangle, |-\rangle\}$ basis.

Answer: The eigenvectors of a nondegenerate Hermitian operator such as O are automatically orthogonal. Since this is a 2×2 vector space, there is just one normalized vector $|\varphi\rangle$ orthogonal to $|\psi\rangle$ up to a phase factor. This is easy to construct:

$$|\varphi\rangle = \sin\tfrac{\theta}{2}|+\rangle - \cos\tfrac{\theta}{2}\, e^{i\phi}|-\rangle$$

Therefore $O|\psi\rangle = a|\psi\rangle$, $O|\varphi\rangle = b|\varphi\rangle$, and

$$\langle\psi|O|\psi\rangle = a, \quad \langle\varphi|O|\varphi\rangle = b, \quad \langle\psi|O|\varphi\rangle = \langle\varphi|O|\psi\rangle = 0$$

We now need matrix elements such as

$$\langle +|O|+\rangle = \langle +|IOI|+\rangle \quad \text{with} \quad I = |\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|$$

So

$$\begin{aligned} \langle +|O|+\rangle &= \langle +|\left(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|\right) O \left(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|\right)|+\rangle \\ &= \langle +|\psi\rangle a\langle\psi|+\rangle + \langle +|\varphi\rangle b\langle\varphi|+\rangle \end{aligned}$$

and similarly with the other matrix elements. Putting everything together,

$$O \doteq \begin{pmatrix} a\cos^2\frac{\theta}{2} + b\sin^2\frac{\theta}{2} & \frac{1}{2}(a-b)\sin\theta\, e^{-i\phi} \\ \frac{1}{2}(a-b)\sin\theta\, e^{i\phi} & b\cos^2\frac{\theta}{2} + a\sin^2\frac{\theta}{2} \end{pmatrix}$$

2. (50 points) You have an operator corresponding to an observable A whose commutator with the Hamiltonian is a constant:

$$[H, A] = ic$$

where c is a real number. At $t = 0$, the system is in an eigenstate of A, such that

$$A|\psi(0)\rangle = a|\psi(0)\rangle$$

Find $\langle A\,\rangle$ as a function of t. (Don't just find $\langle A\,\rangle$ at $t = 0$; that's trivial.)

Hint: Start by finding $[U(t), A]$, where $U(t)$ is the time evolution operator.

Answer: Using the time evolution operator $U = \exp(-\frac{it}{\hbar}H)$,

$$|\psi(t)\rangle = e^{-\frac{it}{\hbar}H}|\psi(0)\rangle$$

Then, since $U^\dagger = \exp(\frac{it}{\hbar}H) = U^{-1}$,

$$\langle A\rangle = \langle\psi(0)|e^{\frac{it}{\hbar}H}Ae^{-\frac{it}{\hbar}H}|\psi(0)\rangle$$

We have to move A to the other side of U, so we need the commutator $[U, A]$. Start with using $HA = AH + ic$, and finding

$$[H^n, A] = H^nA - AH^n = H^{n-1}AH + icH^{n-1} - AH^n = \cdots = nicH^{n-1}$$

after interchanging A and H n times. So

$$\begin{aligned}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-it}{\hbar}\right)^n H^n, A\right] &= \sum_n \frac{nic}{n!}\left(\frac{-it}{\hbar}\right)^n H^{n-1} = \frac{ct}{\hbar}\sum_m \frac{1}{m!}\left(\frac{-it}{\hbar}\right)^m H^m \\ &= \frac{ct}{\hbar}e^{-\frac{it}{\hbar}H} = \frac{ct}{\hbar}U\end{aligned}$$

So, going back,

$$\langle A\rangle = \langle\psi(0)|U^{-1}UA|\psi(0)\rangle - \langle\psi(0)|U^{-1}U\left(\frac{ct}{\hbar}\right)|\psi(0)\rangle = a - \frac{c}{\hbar}t$$