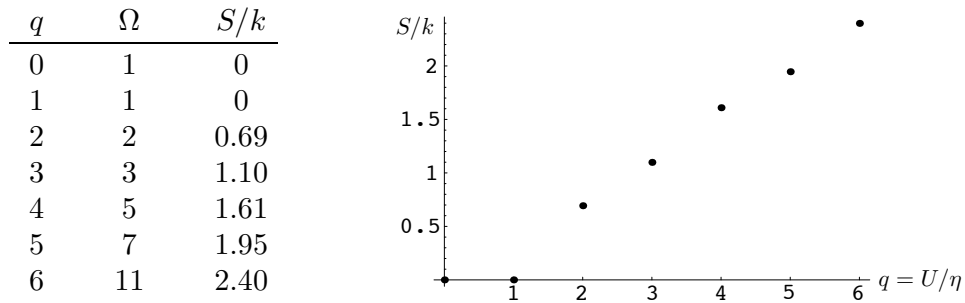


(c) At $\epsilon = \mu$, $\bar{n} \rightarrow \infty$. If N is large, this means $\mu \approx 0$. The plot above fits the Bose-Einstein distribution for the occupancy graph at $kT = 2.2\eta$. Or, you can say that at $\epsilon = \mu + kT$, the occupancy should be $1/(e - 1) \approx 0.6$, and from the data points it looks like this is just above $\epsilon = 2\eta$.

(d) The entropy values turn out to be the same as the fermionic problem, 7.16, I did as an example:



And approximating the slope for $1/kT \approx \Delta S/\Delta U$ again gives $kT \approx 2.2\eta$.

24 The neutron star is a bit simpler than the white dwarf. The only particles are neutrons, and $N = M/m_n$. The kinetic energy is then

$$K = \frac{3}{5}NE_F = \frac{3h^2}{40m_n} \left(\frac{M}{m_n}\right)^{5/3} \left(\frac{9}{4\pi^2 R^3}\right)^{2/3} = \frac{\beta}{R^2}$$

with the constants being packed into β . The gravitational potential energy is the same as the white dwarf, so

$$U = K + V = \frac{\beta}{R^2} - \frac{\alpha}{R}$$

with $\alpha = (3/5)GM^2$. Minimizing the energy with $dU/dR = 0$ gives, for a one solar mass neutron star,

$$R = \frac{2\beta}{\alpha} = (0.093) \frac{h^2}{Gm_n^{8/3} M^{1/3}} = 1.23 \times 10^4 \text{ m}$$

Its density is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = 2.6 \times 10^{17} \text{ kg/m}^3$$

This is comparable to the density of an atomic nucleus. The Fermi temperature is

$$T_F = \frac{1}{k} \frac{h^2}{8m_n} \left(\frac{3N}{\pi V} \right)^{2/3} = \frac{h^2}{8km_n^{5/3}} \left(\frac{9M}{4\pi^2} \right)^{2/3} \frac{1}{R^2} = 6.6 \times 10^{11} \text{ K}$$

This is higher than a white dwarf. In any case, the assumption that $T \ll T_F$ is safe.

A neutron star becomes relativistic and unstable when the average neutron kinetic energy is comparable to $m_n c^2 = 940 \text{ MeV}$. The average $K = \frac{3}{5} kT_F = 34 \text{ MeV}$ for a one solar mass neutron star, which is too small by $1/28$. But since $E_F \propto M^{4/3}$, to get into the relativistic regime, we would need $28^{3/4} = 12$ solar masses. This is a crude calculation, astrophysicists calculate the real critical mass at about 2 to 3 solar masses.

34

- (a) Say $g_{0v} < g_{0c}$. In that case, if μ were to remain constant, there would be fewer electrons added to the conduction band than removed from the valence band. That can't happen, and therefore μ would have to decrease with increasing T . If $g_{0v} > g_{0c}$, μ will have to increase with T .
- (b) The number of conduction electrons is

$$N_c = \int_{E_c}^{\infty} dE g(E) \bar{n}_{FD}(E) = g_{0c} \int_{E_c}^{\infty} dE \frac{\sqrt{E - E_c}}{e^{(E-\mu)/kT} + 1}$$

with $kT \ll E_c - \mu$,

$$N_c \approx g_{0c} \int_{E_c}^{\infty} dE \frac{\sqrt{E - E_c}}{e^{(E-\mu)/kT}} = \frac{g_{0c}(kT)^{3/2}}{e^{(E_c-\mu)/kT}} \int_0^{\infty} dx x^{1/2} e^{-x} = \frac{\sqrt{\pi} g_{0c} (kT)^{3/2}}{2 e^{(E_c-\mu)/kT}}$$

- (c) The occupancy for holes is $1 - \bar{n}_{FD}$. Therefore

$$N_h = \int_{-\infty}^{E_v} dE g(E) [1 - \bar{n}_{FD}(E)] = g_{0v} \int_{-\infty}^{E_v} dE \frac{\sqrt{E_v - E}}{e^{(\mu-E)/kT} + 1}$$

This is much like before. When $kT \ll \mu - E_v$, we get

$$N_h \approx g_{0v} \int_{-\infty}^{E_v} dE \frac{\sqrt{E_v - E}}{e^{(\mu-E)/kT}} = \frac{g_{0v}(kT)^{3/2}}{e^{(\mu-E_v)/kT}} \int_0^{\infty} dx x^{1/2} e^{-x} = \frac{\sqrt{\pi} g_{0v} (kT)^{3/2}}{2 e^{(\mu-E_v)/kT}}$$

(d) $N_h = N_c$ must be true at any T . Therefore,

$$\frac{g_{0c}}{g_{0v}} = \frac{e^{(E_c - \mu)/kT}}{e^{(\mu - E_v)/kT}} = e^{(E_v + E_c - 2\mu)/kT}$$

Taking the log, we find how μ deviates from the center point between the two bands:

$$\mu = \frac{1}{2}(E_v + E_c) + \frac{1}{2}kT \ln(g_{0v}/g_{0c})$$

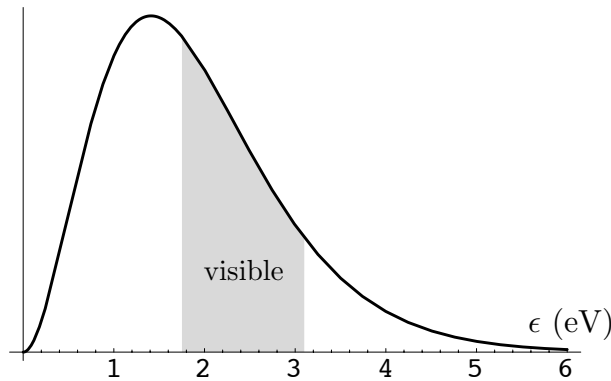
(e) For Si, $\ln(0.44/1.09) = -0.91$. Therefore μ shifts by about $\frac{1}{2} \frac{1}{40}(-0.91) = -0.012$ eV. This is considerably smaller than the band gap of 1.1 eV, so the approximations made should work.

43

(a) Energy density:

$$\frac{U}{V} = \frac{8\pi^5 (kT)^4}{15 (hc)^3} = 0.855 \text{ J/m}^3$$

(b) Spectrum:



(c) Visible light goes from about 1.77 eV to 3.1 eV. With $kT = 0.5$ eV, we have to evaluate

$$\int_{1.77/0.5}^{3.1/0.5} dx \frac{x^3}{e^x - 1}$$

and divide this by the full energy in the spectrum,

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

Done numerically, this results in 0.37.

44

- (a) To get the number of photons, we need to sum the occupancy of each state. With a factor of 2 to account for the two possible photon spin states,

$$N = 2 \sum_{n_x, n_y, n_z} \frac{1}{e^{hcn/2LkT} - 1}$$

As in the text, approximate this as an integral in spherical coordinates (so the eighth of a sphere gives $\pi/2$), and this gives you the desired result. If you then numerically evaluate the integral, you get

$$N = (2.404) 8\pi V \left(\frac{kT}{hc} \right)^3$$

- (b) With equation 7.89, we get $\frac{S}{N} = 3.60 k$
(c) The photon density is

$$\frac{N}{V} = (2.404) 8\pi \left(\frac{kT}{hc} \right)^3$$

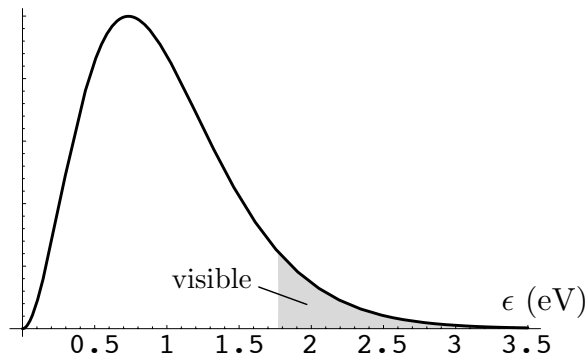
For 300 K, this gives $5.5 \times 10^{14}/\text{m}^3$. For 1500 K, we get $6.8 \times 10^{16}/\text{m}^3$. And for 2.73 K, we end up with $4.1 \times 10^8/\text{m}^3$. The cosmic microwave photon density in the universe is much larger than the density of matter overall.

51

- (a) The power $\mathcal{P} = e\sigma AT^4$, so

$$A = \frac{\mathcal{P}}{e\sigma T^4} = 6.6 \times 10^{-5} \text{ m}^2$$

- (b) The spectrum has a peak at $E = 2.82 kT = 0.73$ eV. The wavelength is $\lambda = hc/E = 1.7 \times 10^{-6}$ m, in the infrared.
- (c) Plotting, we see that most of the energy is radiated away in the infrared.



- (d) Change variables to $x = E/kT$. At 3000 K, $kT = 0.26$ eV. The visible spectrum is between $1.77/0.26 = 6.8$ and $3.1/0.26 = 11.9$. The energy in the visible is

$$\int_{6.8}^{11.9} dx \frac{x^3}{e^x - 1}$$

This needs to be evaluated numerically. All the energy is

$$\int_0^{\infty} dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

Their ratio is the visible energy fraction, which is 0.083. Not very efficient.

- (e) Increasing efficiency requires shifting the spectrum peak toward the visible, which means a higher temperature.
- (f) The efficiency requires figuring out the integral ratio in (d) as a function of T , and either numerically differentiating it for a maximum, or plotting it and visually estimating the peak, around 7000 K.

53

- (a) Putting together the equations provided with the spectrum peak being at $E = 2.82 kT$ results in $\lambda = 8.4 \times 10^4$ m.

-
- (b) Total power for a blackbody:

$$\mathcal{P} = \sigma AT^4 = 9 \times 10^{-29} \text{ W} = 6 \times 10^{-10} \text{ eV/s}$$

This is really low.

- (c) The energy lost from the black hole must come from a decrease in mass:
 $\mathcal{P} = -d(Mc^2)/dt$. Therefore

$$\frac{dM}{dt} = -\frac{\sigma AT^4}{c^2} = -\frac{H}{M^2} \quad \text{with} \quad H = 4.0 \times 10^{15} \text{ kg}^3/\text{s}$$

This gives $M^2 dM = -H dt$, which we can integrate to get the lifetime τ :

$$\tau = \int_0^\tau dt = -\frac{1}{H} \int_{M_i}^0 dM M^2 = \frac{M_i^3}{3H}$$

- (d) For a solar mass, $\tau = 7 \times 10^{74}$ s; about 10^{57} times the age of the universe.
- (e) The age of the universe is about 4×10^{17} s. The initial mass of a black hole with that lifetime would be 1.7×10^{11} kg, smaller than the sun by about a factor of 10^{19} . The peak of its spectrum would be at about $\lambda = 7 \times 10^{-15}$ m, an extremely energetic gamma ray.

64

- (a) The number of magnons is

$$N_m = \sum_{n_x, n_y, n_z} \frac{1}{e^{E/kT} - 1}$$

As usual, $E = \hbar^2 k^2 / 2m^* = \hbar^2 n^2 / 8m^* L^2$. Again, as usual, convert the sum to an integral using spherical coordinates, ending up with

$$N_m = \frac{\pi}{2} \int_0^\infty dn \frac{n^2}{e^{n^2 \hbar^2 / 8m^* L^2 kT} - 1}$$

As usual, change variables to $x = E/kT$, and after some algebra, you get

$$N_m = 2\pi V \left(\frac{2m^* kT}{h^2} \right)^{3/2} \int_0^\infty dx \frac{x^{1/2}}{e^x - 1} = 2\pi V \left(\frac{2m^* kT}{h^2} \right)^{3/2} \quad (2.315)$$

-
- (b) Each magnon reduces the magnetization by $2\mu_B$. Therefore the fractional reduction in magnetization is

$$\frac{N_m}{N} = \left(\frac{T}{T_0}\right)^{3/2} \quad \text{where} \quad T_0 = (0.0839) \frac{h^2}{m^*k} \left(\frac{N}{V}\right)^{2/3}$$

For iron, $T_0 = 4150$ K.

- (c) The energy is

$$\begin{aligned} U &\approx \frac{\pi}{2} \int_0^\infty dn n^2 \frac{1}{e^{n^2 h^2 / 8m^*L^2 kT} - 1} \frac{h^2 n^2}{8m^*L^2} \\ &= 2\pi V \left(\frac{2m^*kT}{h^2}\right)^{3/2} (kT) \int_0^\infty dx \frac{x^{3/2}}{e^x - 1} \\ &= 2\pi V \left(\frac{2m^*kT}{h^2}\right)^{3/2} kT (1.783) = (31.69) kV \left(\frac{km^*}{h^2}\right)^{3/2} T^{5/2} \end{aligned}$$

$$C_V = \frac{\partial U}{\partial T} = (31.69) \frac{5}{2} kV \left(\frac{km^*T}{h^2}\right)^{3/2}$$

Therefore

$$\frac{C_V}{Nk} = \left(\frac{T}{T_1}\right)^{3/2} \quad \text{with} \quad T_1 = \frac{h^2}{km^*} \left(\frac{N}{V}\right)^{2/3} \left(\frac{2}{5(31.69)}\right)^{2/3} = 0.646 T_0$$

For iron, this is $T_1 = 2680$ K, which means the magnon contribution is small at room temperatures and below. But when the temperature is low enough, the magnons will contribute more than the phonons, since the phonon $C_V \propto T^3$.

- (d) The only difference here is that the angular integral is over a quarter-circle, and the radial integral involves $dn n$ rather than $dn n^2$. After changing variables to $x \propto n^2$ (since $E \propto n^2$), this leaves us with an integral like

$$N_m \propto \int_0^\infty dx \frac{1}{e^x - 1} = \infty$$

66

(a) The ground state has

$$E_0 = \frac{h^2}{8mL^2}(1^2 + 1^2 + 1^2) = 1.14 \times 10^{-32} \text{ J} = 7.1 \times 10^{-14} \text{ eV}$$

(b) Using equation 7.126, $kT_c = (0.224)N^{2/3}E_0$. In that case,

$$\frac{kT_c}{E_0} = (0.224)(10000)^{2/3} = 104 \quad \Rightarrow \quad T_c = 8.6 \times 10^{-8} \text{ K}$$

(c) At $T = 0.9T_c$,

$$N_0 = \left[1 - \left(\frac{T}{T_c} \right)^{3/2} \right] N = 0.146 N = 1460$$

Then, with equation 7.120, we get $E_0 - \mu = kT/N_0 = 4.6 \times 10^{-15} \text{ eV}$. The first excited state have energies of $2E_0$ each. The occupancy of each is

$$N_1 = \frac{1}{e^{(E_1 - \mu)/kT} - 1} = 87$$

(d) With 10^6 atoms, T_c will be higher by a factor of $(10^6/10^4)^{2/3} = 21.5$, so $T_c = 1.85 \times 10^{-6} \text{ K}$. The fraction in the ground state remains the same, but the actual number is large by a factor of 100. Therefore $E_0 - \mu = 9.8 \times 10^{-16} \text{ eV}$, and $N_1 = 1990$. It looks like, if T/T_c is kept the same, N_0/N_1 increases with N .