Homework Solutions #8 (McIntyre Chapter 9)

6 Look at the matrix elements of the commutator for $\bar{a}$ and $\bar{a}^\dagger$, where $\bar{a}|n\rangle = |n-1\rangle$ and $\bar{a}^\dagger|n\rangle = |n+1\rangle$:

$$\langle m| \left[\bar{a}, \bar{a}^\dagger\right] |n\rangle = \langle m| \left(\bar{a}\bar{a}^\dagger - \bar{a}^\dagger\bar{a}\right) |n\rangle = \langle m|n\rangle - \langle m|n\rangle = 0$$

Since the matrix elements of the commutator are all zero, $[\bar{a}, \bar{a}^\dagger] = 0$.

Such operators do not exist for the simple harmonic oscillator, because there are no negative n states that could be generated from the ground state by $\bar{a}|0\rangle = |-1\rangle$ etc.

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(a) The ground state $\varphi_0 = N_0 e^{-m\omega x^2/2\hbar}$, which is a real, even function. Therefore

$$\langle x\rangle = \int dx\, \varphi_0^* x \varphi_0 = 0 \qquad \langle p\rangle = -i\hbar \int dx\, \varphi_0^* \frac{\partial}{\partial x}\varphi_0 = 0$$

The probability density is Gaussian with standard deviation $\sigma = \sqrt{\hbar/2m\omega}$. So

$$\langle x^2\rangle = \int dx\, \varphi_0^* x^2 \varphi_0 = \int dx\, N_0^2 e^{-m\omega x^2/\hbar} x^2 = \frac{\hbar}{2m\omega}$$

$$\begin{aligned}\langle p^2\rangle &= \int dx\, \varphi_0^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right)\varphi_0 = \hbar^2 \int dx\, N_0^2 e^{-m\omega x^2/\hbar}\left(\frac{m\omega}{\hbar} - \frac{m^2\omega^2}{\hbar^2}x^2\right) \\ &= m\omega\hbar - m^2\omega^2\langle x^2\rangle = \frac{m\omega\hbar}{2}\end{aligned}$$

(b) For all $n = 0, 1, 2, \ldots$

$$\langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, \langle n| \left(a^\dagger + a\right)|n\rangle = 0$$

$$\langle n|p|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\, \langle n| \left(a^\dagger - a\right)|n\rangle = 0$$

$$\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\, \langle n| \left((a^\dagger)^2 + a^2 + 2N + 1\right)|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

$$\langle n|p^2|n\rangle = -\frac{\hbar m\omega}{2}\, \langle n| \left((a^\dagger)^2 + a^2 - 2N - 1\right)|n\rangle = \hbar m\omega\left(n + \frac{1}{2}\right)$$

(c) For all n,

$$\Delta x \Delta p = \hbar \left(n + \frac{1}{2}\right) \geq \frac{\hbar}{2}$$

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(a) With the given probabilities, the state must be

$$|\psi(t)\rangle = 0.6\, e^{-i\omega_1 t}|1\rangle + 0.8\, e^{-i\omega_2 t} e^{i\theta}|2\rangle$$

We need to determine the relative phase θ by finding $\langle x \rangle$.

$$\begin{aligned}
\langle x \rangle &= \langle \psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\langle \psi(t)|a^\dagger|\psi(t)\rangle + \langle \psi(t)|a|\psi(t)\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(0.8\, e^{i\omega_2 t} e^{-i\theta} \sqrt{2}\, 0.6\, e^{-i\omega_1 t} + 0.8\, e^{-i\omega_2 t} e^{i\theta} \sqrt{2}\, 0.6\, e^{i\omega_1 t}\right) \\
&= 0.96 \sqrt{\frac{\hbar}{m\omega}} \cos(\omega t - \theta)
\end{aligned}$$

For this to be minimum at $t = 0$, we must have $\cos\theta = -1$, and so $\theta = \pi$. The state is

$$|\psi(t)\rangle = 0.6\, e^{-i\omega_1 t}|1\rangle - 0.8\, e^{-i\omega_2 t}|2\rangle$$

and the wave function is

$$\begin{aligned}
\psi(t) &= 0.6\, e^{-i\omega_1 t}\varphi_1 - 0.8\, e^{-i\omega_2 t}\varphi_2 \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} \left[0.849\, e^{i\frac{3}{2}\omega t} \sqrt{\frac{m\omega}{\hbar}} x - 0.566\, e^{i\frac{5}{2}\omega t} \left(\frac{2m\omega}{\hbar} x^2 - 1\right)\right]
\end{aligned}$$

(b) Momentum:

$$\begin{aligned}
\langle p \rangle &= \langle \psi(t)|p|\psi(t)\rangle = i\sqrt{\frac{\hbar m\omega}{2}} \left(\langle \psi(t)|a^\dagger|\psi(t)\rangle - \langle \psi(t)|a|\psi(t)\rangle\right) \\
&= -i\sqrt{\frac{\hbar m\omega}{2}} \left(0.8\, e^{i\omega_2 t} \sqrt{2}\, 0.6\, e^{-i\omega_1 t} - 0.8\, e^{-i\omega_2 t} \sqrt{2}\, 0.6\, e^{i\omega_1 t}\right) \\
&= 0.96 \sqrt{\hbar m\omega} \sin(\omega t)
\end{aligned}$$

(c) The energy expectation value is straightforward, since it is not time-dependent.

$$\langle E \rangle = 0.36\, E_1 + 0.64\, E_2 = 2.14\, \hbar\omega$$

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$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad H \doteq \hbar\omega \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 0 & \cdots \\ 0 & 0 & 27 & 0 & \cdots \\ 0 & 0 & 0 & 64 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\langle m|A|n\rangle = 3n^2 \delta_{m,n+2} \quad \Rightarrow \quad A \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 0 & \cdots \\ 0 & 12 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$