Homework Solutions #9 (McIntyre Ch. 10)

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$$\begin{aligned} H' = \gamma x^3 &= \gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2} \left(a^\dagger + a\right)^3 \\ &= \gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2} \left(a^{\dagger 3} + a^3 + 3a^{\dagger 2}a + 3a^\dagger a^2 + 3a^\dagger + 3a\right) \end{aligned}$$

(a) The first order corrections are

$$E_n^{(1)} = \langle n|H'|n\rangle = 0$$

since every term in $\left(a^\dagger + a\right)^3$ will change n in the right-side $|n\rangle$.

(b) For the second-order corrections, we need a number of off-diagonal matrix elements:

$$\langle 0|H'|1\rangle = \langle 1|H'|0\rangle^* = 3\,\gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

$$\langle 0|H'|2\rangle = \langle 2|H'|0\rangle^* = 0$$

$$\langle 0|H'|3\rangle = \langle 3|H'|0\rangle^* = \sqrt{6}\,\gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

$$\langle 1|H'|2\rangle = \langle 2|H'|1\rangle^* = 6\sqrt{2}\,\gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

$$\langle 1|H'|3\rangle = \langle 3|H'|1\rangle^* = 0$$

$$\langle 1|H'|4\rangle = \langle 4|H'|1\rangle^* = 2\sqrt{6}\,\gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

$$\langle 2|H'|3\rangle = \langle 3|H'|2\rangle^* = 9\sqrt{3}\,\gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

$$\langle 2|H'|4\rangle = \langle 4|H'|2\rangle^* = 0$$

$$\langle 2|H'|5\rangle = \langle 5|H'|2\rangle^* = 2\sqrt{15}\,\gamma \left(\frac{\hbar}{2m\omega}\right)^{3/2}$$

All other matrix elements that go into the corrections $E_0^{(2)}$, $E_1^{(2)}$, and $E_2^{(2)}$ are zero. We now add up the terms for the corrections:

$$\begin{aligned} E_0^{(2)} &= \sum_{m\neq 0} \frac{|\langle m|H'|0\rangle|^2}{(0-m)\hbar\omega} = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{1}{\hbar\omega}\left(-\frac{3^2}{1} - \frac{6}{3}\right) \\ &= -\gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{11}{\hbar\omega} \end{aligned}$$

$$\begin{aligned} E_1^{(2)} &= \sum_{m\neq 1} \frac{|\langle m|H'|1\rangle|^2}{(1-m)\hbar\omega} = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{1}{\hbar\omega}\left(\frac{3^2}{1} - \frac{6^2 2}{1} - \frac{2^2 6}{3}\right) \\ &= -\gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{71}{\hbar\omega} \end{aligned}$$

$$\begin{aligned} E_2^{(2)} &= \sum_{m\neq 2} \frac{|\langle m|H'|2\rangle|^2}{(2-m)\hbar\omega} = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{1}{\hbar\omega}\left(\frac{6^2 2}{1} - \frac{9^2 3}{1} - \frac{2^2 15}{3}\right) \\ &= -\gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{191}{\hbar\omega} \end{aligned}$$

(c) Use the same matrix elements for the eigenstate corrections.

$$|0^{(1)}\rangle = \sum_{m\neq 0} \frac{\langle m|H'|0\rangle}{(0-m)\hbar\omega}|m\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \frac{\gamma}{\hbar\omega}\left(-3|1\rangle - \frac{\sqrt{6}}{3}|3\rangle\right)$$

$$|1^{(1)}\rangle = \sum_{m\neq 1} \frac{\langle m|H'|1\rangle}{(1-m)\hbar\omega}|m\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \frac{\gamma}{\hbar\omega}\left(3|0\rangle - 6\sqrt{2}|2\rangle - \frac{2\sqrt{6}}{3}|4\rangle\right)$$

$$\begin{aligned} |2^{(1)}\rangle &= \sum_{m\neq 2} \frac{\langle m|H'|2\rangle}{(2-m)\hbar\omega}|m\rangle \\ &= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \frac{\gamma}{\hbar\omega}\left(6\sqrt{2}|1\rangle - 9\sqrt{3}|3\rangle - \frac{2\sqrt{15}}{3}|5\rangle\right) \end{aligned}$$

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(a) First-order energy corrections:

$$\begin{aligned} E_n^{(1)} &= \langle n|H'|n\rangle = \frac{2}{L}\int_0^L dx\, \sin^2\left(\frac{n\pi}{L}x\right) LV_0\,\delta\left(x-\frac{L}{2}\right) \\ &= 2V_0 \sin^2\left(\frac{n\pi}{2}\right) = \begin{cases} 2V_0 & \text{odd } n \\ 0 & \text{even } n \end{cases} \end{aligned}$$

(b) For odd n, the energy eigenfunctions $\varphi_n(\frac{L}{2}) \neq 0$, so the perturbation has an effect. But for even n, $\varphi_n(\frac{L}{2}) = 0$.

(c) The largest contribution is from the state $|n\rangle$ where $n > 1$ and

$$\left|\frac{\langle n|H'|1\rangle}{E_1 - E_n}\right| \text{ is the largest.}$$

$$\begin{aligned} \langle n|H'|1\rangle &= \frac{2}{L}\int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{\pi}{L}x\right) LV_0\,\delta\left(x-\frac{L}{2}\right) \\ &= 2V_0 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Therefore the largest contribution comes from $|3\rangle$.

(d) The correction is

$$\frac{2V_0}{\varepsilon L}\int_{\frac{L}{2}-\frac{\varepsilon L}{2}}^{\frac{L}{2}+\frac{\varepsilon L}{2}} dx\, \sin^2\left(\frac{n\pi}{L}x\right) = V_0\left(1 + \frac{\sin \pi\varepsilon}{\pi\varepsilon}\right)$$

(e) At the limit,

$$\lim_{\varepsilon\to 0} V_0\left(1 + \frac{\sin\pi\varepsilon}{\pi\varepsilon}\right) = 2V_0$$

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(a) When $\varepsilon = 0$, H is diagonal, so the eigenvalues ($E_1^{(0)} = 3V_0$, $E_2^{(0)} = 3V_0$, $E_3^{(0)} = 5V_0$, $E_4^{(0)} = 7V_0$) and eigenvectors are trivial.

(b) Since the diagonal elements of H' are zero, the first-order energy corrections for the nondegenerate energy levels are zero:

$$E_3^{(1)} = \langle 3|H'|3\rangle = 0 \qquad E_4^{(1)} = \langle 4|H'|4\rangle = 0$$

The second-order corrections for these are

$$E_3^{(2)} = \sum_{m\neq 3} \frac{|\langle m|H'|3\rangle|^2}{5V_0 - E_m^{(0)}} = \left(\frac{2^2}{2} - \frac{1}{2}\right)\varepsilon^2 V_0 = \frac{3}{2}\varepsilon^2 V_0$$

$$E_4^{(2)} = \sum_{m\neq 4} \frac{|\langle m|H'|4\rangle|^2}{7V_0 - E_m^{(0)}} = \frac{1}{2}\varepsilon^2 V_0$$

Diagonalizing H' in the degenerate subspace,

$$\begin{vmatrix} -\lambda & \varepsilon V_0 \\ \varepsilon V_0 & -\lambda \end{vmatrix} = \lambda^2 - (\varepsilon V_0)^2 = 0 \quad \Rightarrow \quad \lambda = \pm\varepsilon V_0$$

Therefore

$$E_1^{(1)} = \varepsilon V_0 \qquad E_2^{(1)} = -\varepsilon V_0$$

(which is which is arbitrary.)