Solutions to Third Midterm

1. (40 points) Find the series solutions to

$$x^2y'' + x^2y' - 2y = 0$$

Write down the recurrence relationships and the first two nonzero terms for the series you find. Do any of your solutions terminate (is a finite series)?

Answer: Try $y = \sum a_n x^{n+s}$ and substitute it in the equation:

$$\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s} + \sum_{n=0}^{\infty} a_n(n+s)x^{n+s+1} - \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0$$

The second sum needs to be shifted. We get

$$\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s} + \sum_{n=1}^{\infty} a_{n-1}(n+s-1)x^{n+s} - \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0$$

First, we deal with the special case of $n = 0$:

$$a_0 s(s-1) - 2a_0 = 0 \qquad \Rightarrow \qquad s^2 - s - 2 = 0$$

This gives two solutions: $s = -1$ and $s = 2$.

The $n > 0$ terms give a recurrence relationship,

$$a_n = -\frac{n+s-1}{(n+s)(n+s-1)-2} a_{n-1}$$

For $s = -1$, this is

$$a_n = -\frac{n-2}{(n-1)(n-2)-2} a_{n-1}$$

But we see that this series will terminate at $n = 2$, so that $a_n = 0$ for $n \geq 2$. We have a non-zero $a_1 = -\frac{1}{2}a_0$. This means one solution is

$$y_1 = A(x^{-1} - x^0/2) = A\left(\frac{1}{x} - \frac{1}{2}\right)$$

with A an undetermined constant.

For $s = 2$, we get an infinite series with

$$a_n = -\frac{n+1}{(n+2)(n+1)-2} a_{n-1}$$

This leads to

$$y_2 = B\left(x^2 - \frac{1}{2}x^3 + \frac{3}{20}x^4 - \ldots\right)$$

2. (20 points)

(a) Use $j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$ and $\frac{d}{dx}J_p(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$ to show that

$$\frac{d}{dx}j_n(x) = \frac{n}{x}j_n(x) - j_{n+1}(x)$$

(b) Use the result of part (a) and $j_0(x) = \frac{\sin x}{x}$ to evaluate

$$\int_0^{\pi/2} dx\, j_1(x)$$

Answer:

(a)

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{d}{dx}\left[x^{-\frac{1}{2}} J_{n+\frac{1}{2}}(x)\right] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[-\frac{1}{2}x^{-\frac{3}{2}} J_{n+\frac{1}{2}}(x) + x^{-\frac{1}{2}} J'_{n+\frac{1}{2}}(x)\right] =$$

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[-\frac{1}{2}x^{-\frac{3}{2}} J_{n+\frac{1}{2}}(x) + x^{-\frac{1}{2}} \frac{n+\frac{1}{2}}{x} J_{n+\frac{1}{2}}(x) - x^{-\frac{1}{2}} J_{n+\frac{3}{2}}(x)\right] =$$

$$-\frac{1}{2x}j_n(x) + \frac{n+\frac{1}{2}}{x}j_n(x) - j_{n+1}(x) = \frac{n}{x}j_n(x) - j_{n+1}(x)$$

(b) Use the previous result with $n = 0$, giving $j_1(x) = -\frac{d}{dx}j_0(x)$. So

$$\int_0^{\pi/2} dx\, j_1(x) = -\int_0^{\pi/2} dx\, \frac{d}{dx}j_0(x) = j_0(x)\big|_{\frac{\pi}{2}}^{0} = \left.\frac{\sin x}{x}\right|_{\frac{\pi}{2}}^{0} = 1 - \frac{2}{\pi}$$

3. (40 points) You have a partial differential equation

$$\left(x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + x^2 + \frac{\partial}{\partial t}\right) f(x,t) = 0$$

Say you are looking for basis functions b_α to express $f(x,t)$, where $b_\alpha(x,t) = X(x)T(t)$ and α is a parameter. You also have the conditions that

$$f(0,t) < \infty \qquad \text{and} \qquad f(x,\infty) < \infty$$

Find the functions b_α and any restrictions on the values of α.

Answer: Putting $b = XT$ into the PDE, and dividing the whole equation by XT, we get

$$\frac{1}{X}\left(x^2X'' + x\,X' + x^2X\right) + \frac{\dot{T}}{T} = 0$$

Notice that the two terms depend only on x or only on t. Therefore they must be constants. Set

$$\frac{1}{X}\left(x^2X'' + x\,X' + x^2X\right) = \alpha \qquad \text{and} \qquad \frac{\dot{T}}{T} = -\alpha$$

Do the temporal part first. We get

$$\dot{T} + \alpha T = 0 \qquad \Rightarrow \qquad T = A\,e^{-\alpha t}$$

Now, since $b(x,\infty) < \infty$, we must have $\alpha \geq 0$.

The spatial equation is

$$x^2X'' + x\,X' + (x^2 - \alpha)X = 0$$

This is a Bessel equation with $p^2 = \alpha$. Therefore

$$X = B\,J_{\sqrt{\alpha}}(x) + C\,N_{\sqrt{\alpha}}(x)$$

Since $b(0,t) < \infty$, we have to set $C = 0$. The solutions are, finally,

$$b(x,t) = J_{\sqrt{\alpha}}(x)\,e^{-\alpha t}$$

with $\alpha \geq 0$.