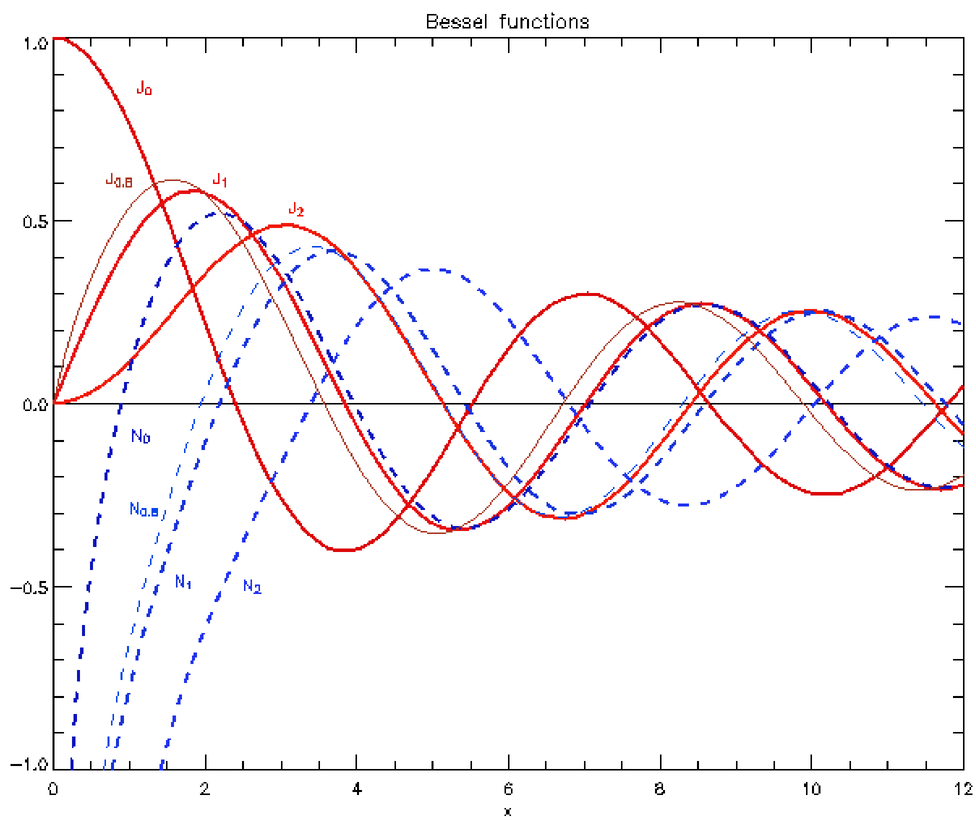
Homework Solutions, Chapter 12

Special Here's an IDL plot of J_0, $J_{0.8}$, J_1, J_2 (solid lines), and N_0, $N_{0.8}$, N_1, N_2 (dotted lines)



11.7 Plugging $y = \sum a_n x^{n+s}$ and its derivatives into the ODE, we get

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} - \sum_{n=0}^{\infty} 2a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

Rearranging the indices,

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} - \sum_{n=0}^{\infty} 2a_n x^{n+s} - \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0$$

First, deal with $n = 0$, which gives

$$[s(s-1)-2]a_0 = (s-2)(s+1)a_0 = 0$$

Since $a_0 \neq 0$, this means $s = 2$ or $s = -1$. The $n = 1$ term gives

$$(s^2 + s - 2)a_1 = 0$$

For both $s = 2$ and $s = -1$, the only way to satisfy this is $a_1 = 0$.

For $s = 2$, the recurrence relationship obtained from $n \geq 2$ is

$$[(n+s)(n+s-1)-2]\, a_n - a_{n-2} = 0 \Rightarrow a_n = \frac{a_{n-2}}{n(n+3)}$$

This is good enough to define a solution; the arbitrary constant is a_0. Since $a_1 = 0$, $a_m = 0$ for all odd m. The first few terms of this solution is

$$y_1(x) = a_0\, x^2 \left[1 + \frac{1}{10}x^2 + \frac{1}{280}x^4 + \dots\right]$$

For $s = -1$, the recurrence relationship obtained from $n \geq 2$ is

$$[(n+s)(n+s-1)-2]\, a_n - a_{n-2} = 0 \Rightarrow a_n = \frac{a_{n-2}}{n(n-3)}$$

Again, $a_m = 0$ for all odd m. The first few terms of this solution is (calling the arbitrary constant b_0 to distinguish it from the previous a_0)

$$y_2(x) = \frac{b_0}{x}\left[1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \dots\right]$$

12.9 Write out the series for

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sqrt{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\frac{1}{2})!}\left(\frac{x}{2}\right)^{2n+1}$$

Notice that

$$(n+\frac{1}{2})! = (n+\frac{1}{2})(n-\frac{1}{2})\dots\frac{1}{2}\Gamma(\frac{1}{2})$$

This is a product of halves of odd numbers going down from $2n+1$. So, using the "!!" notation, $(n+\frac{1}{2})! = \sqrt{\pi}(2n+1)!!/2^{n+1}$. Now look at $n!$;

$$n! = n(n-1)\dots 1 = \left(\frac{2n}{2}\right)\left(\frac{2(n-1)}{2}\right)\dots\frac{2}{2} = \frac{(2n+1)!}{2^n(2n+1)!!}$$

In other words, if $(2n+1)!!$ gives a product of odd numbers, the missing even numbers in $(2n+1)!$ can be obtained from $n!$. Putting this together,

$$n!(n+\frac{1}{2})! = \frac{(2n+1)!}{2^{2n+1}}\sqrt{\pi}$$

Now, go back to the series:

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sqrt{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!\sqrt{\pi}} \frac{x^{2n+1}}{2^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x$$

15.6 Where x_m is an extrema of $J_p(x)$, $J_p'(x_m) = 0$. Use the recurrence relationship

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x) \Rightarrow J_{p-1}(x_m) = J_{p+1}(x_m)$$

When $J_p(x_0) = 0$, another recurrence relationship gives

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x) \Rightarrow J_{p-1}(x_0) = -J_{p+1}(x_0)$$

Then, using the first relationship once more, $2J_{p-1}(x_0) = 2J_p'(x_0)$, so $J_{p-1}(x_0) = J_p'(x_0)$.

19.3 For $p = n + \frac{1}{2}$, we have

$$\int_0^1 dx\, x J_{n+1/2}(ax) J_{n+1/2}(bx) = \frac{2(ab)^{\frac{1}{2}}}{\pi} \int_0^1 dx\, x^2 j_n(ax) j_n(bx)$$

Now, the other side. Doing the derivative,

$$\frac{d}{dx} J_{n+1/2}(x) = \frac{d}{dx}\left(\sqrt{\frac{2x}{\pi}}\, j_n(x)\right) = \sqrt{\frac{2}{\pi}}\left(\sqrt{x}\, j_n'(x) + \frac{1}{2\sqrt{x}}\, j_n(x)\right)$$

But noticing that $j_n(a) = 0$ (because a is a zero of the Bessel function), then

$$\frac{1}{2}\, J_{n+1/2}'^{2}(a) = \frac{a}{\pi}\, j_n'^{2}(a)$$

So

$$\int_0^1 dx\, x^2 j_n(ax) j_n(bx) = \begin{cases} 0 & \text{if } a \neq b \\ \frac{1}{2} j_n'^{2}(a) & \text{if } a = b \end{cases}$$

23.10 Use $\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x)$ to get the integral:

$$I = \int_0^\infty dx\, x^{-p}J_{p+1}(x) = -\left. x^{-p}J_p(x)\right|_0^\infty$$

$x^{-p}J_p(x) \to 0$ at $x \to \infty$. So we need the limit as $x \to 0$. The first term in the series expansion for $x^{-p}J_p(x)$ goes like x^0, and the higher powers of x all give 0. The $n = 0$ term gives

$$I = \frac{(-1)^0}{0!(0+p)!}\frac{1}{2^{0+p}} = \frac{1}{2^p p!}$$