Homework Solutions, Chapter 13

4.1 (Special) To get b_n:

$$b_n = \frac{2}{L}\int_0^L dx\,\sin\frac{n\pi}{L}x\,f(x)$$

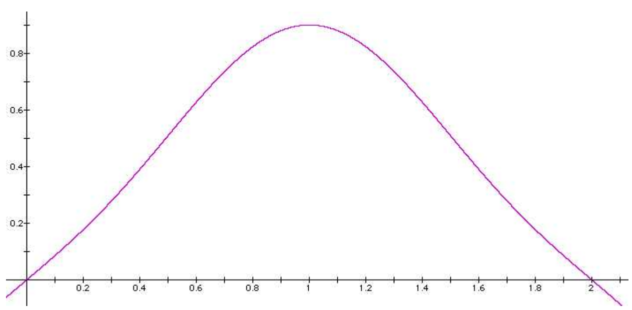
Changing variables to $u = n\pi x/L$, we get

$$b_n = \frac{4h}{n^2\pi^2}\left[\int_0^{\frac{n\pi}{2}} du\,u\sin u + \int_{\frac{n\pi}{2}}^{n\pi} du\,(n\pi - u)\sin u\right] = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

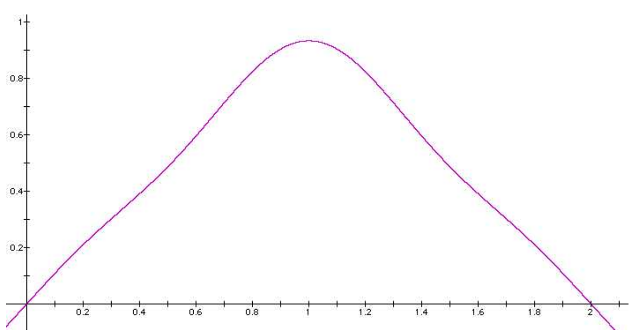
Therefore all $b_n = 0$ for even n, and signs alternate otherwise:

$$b_1 = \frac{8h}{\pi^2}, \quad b_3 = -\frac{8h}{9\pi^2}, \quad b_5 = \frac{8h}{25\pi^2}, \quad \ldots$$

Here are the plots of the sums of the Fourier series for $f(x)$ up to $n = 3$, where I have taken $L = 2$ and $h = 1$ (it doesn't matter).

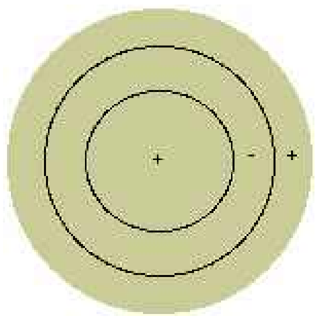


and up to $n = 5$,

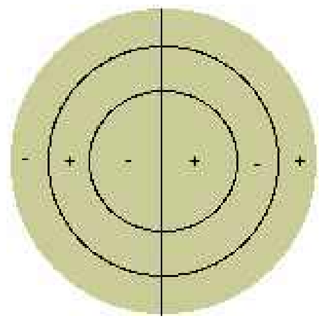


Notice how they approach the triangular shape of $f(x)$.

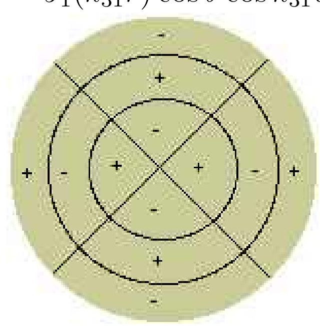
6.1 Nodes are where the Bessel J functions have zeros, or where the angular part, $\cos n\theta$ (or $\sin n\theta$, this just rotates the graph) is zero. From your previous graphs of J_0, J_1, J_2, you know that the radial function will cross zero $m-1$ times, excluding that at $r=0$. This means you should draw $m-1$ concentric circles inside the drumhead. The angular part crosses zero at $\theta = (2j+1)\pi/2n$, where j is an integer; i.e. at odd multiples of $\pi/2n$. So divide the circle of the drumhead accordingly:



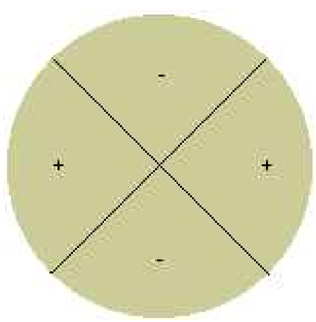
$z = J_0(k_{30}r)\cos k_{30}vt$



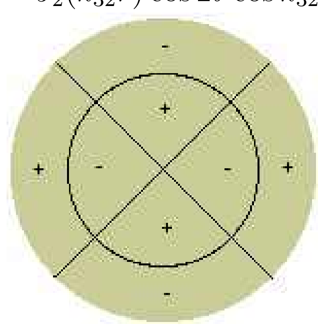
$z = J_1(k_{31}r)\cos\theta\cos k_{31}vt$



$z = J_2(k_{32}r)\cos 2\theta\cos k_{32}vt$



$z = J_2(k_{12}r)\cos 2\theta\cos k_{12}vt$



$z = J_2(k_{22}r)\cos 2\theta\cos k_{22}vt$

6.3 Take $z(x,y,t) = X(x)Y(y)T(t)$. We get

$$YT\frac{d^2X}{dx^2} + XT\frac{d^2Y}{dy^2} = \frac{XY}{v^2}\frac{d^2T}{dt^2}$$

giving

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{v^2T}\frac{d^2T}{dt^2}$$

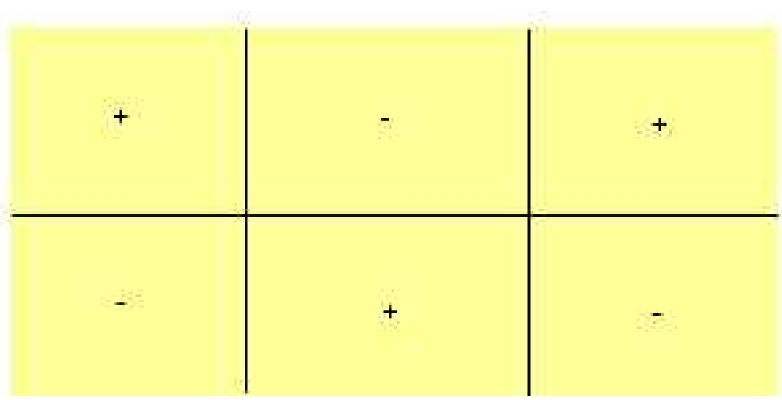
This will be satisfied if each term is a constant. Call the constant from the T-term $-k^2$. Then the X and Y terms give

$$-k_x^2 - k_y^2 = -k^2 \qquad \Rightarrow \qquad \frac{\omega^2}{v^2} = k_x^2 + k_y^2 \tag{1}$$

All the resulting differential equations in x, y and t are of the form $f''+k^2f = 0$, which give sines and cosines. The boundary conditions demand $X(0) = 0$. This rules out the $\cos k_x x$ solution. Then we have $X(a) = 0$, which means $\sin k_x a = 0$, so $k_x = n\pi/a$ for positive integer n. An identical argument works for the Y's, restricting k_y to $m\pi/b$. So the normal modes are defined by the integers n, m. The characteristic angular frequencies are determined by the constraint (1). Recalling $\omega = 2\pi\nu$, we get

$$\nu_{nm} = \frac{v}{2\pi}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} = \frac{v}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

The nodal lines are trivial; for each m, n pair there are $n-1$ equally spaced vertical lines and $m-1$ horizontal lines; e.g.:



$z = \sin(3\pi x/a)\sin(2\pi y/b)\cos\omega_{32}t$

If $a = b$, all frequencies with $n \neq m$ are degenerate, because then $\nu_{mn} = \nu_{nm}$.

10.25 Take $u(x,t) = X(x)T(t)$. This results in

$$T\frac{d^2X}{dx^2} = \frac{X}{v^2}\frac{d^2T}{dt^2} + \lambda^2 XT \qquad \Rightarrow \qquad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2T}\frac{d^2T}{dt^2} + \lambda^2$$

To satisfy this, we must have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \qquad \text{and} \qquad \frac{1}{v^2T}\frac{d^2T}{dt^2} = -k^2 - \lambda^2$$

The X-equation gives the usual $X = \sin n\pi x/l$ solutions, and the T-equation gives $\sin \omega t$ where $\omega^2 = (2\pi\nu)^2 = k^2v^2 + \lambda^2$. Replacing $k = n\pi/l$ here, with some algebra, gives

$$\nu_n = \frac{v}{2}\sqrt{\left(\frac{n}{l}\right)^2 + \left(\frac{\lambda}{\pi}\right)^2}$$