Solutions to Final

1. (35 points) You have a set of three basis vectors

$$\{|e_1\rangle, |e_2\rangle, |e_3\rangle\} \qquad \langle e_n|e_m\rangle = \delta_{nm}, \quad \forall\, n, m$$

You also have a linear operator $\hat{A}$ such that

$$\begin{aligned} \hat{A}|e_1\rangle &= 2|e_1\rangle + 2i|e_2\rangle + 2|e_3\rangle \\ \hat{A}|e_2\rangle &= -2i|e_1\rangle + 5|e_2\rangle + i|e_3\rangle \\ \hat{A}|e_3\rangle &= 2|e_1\rangle - i|e_2\rangle + 5|e_3\rangle \end{aligned}$$

Find the eigenvalues of $\hat{A}$, and an *orthonormal* set of eigenvectors of $\hat{A}$ corresponding to these eigenvalues.

Write each of your eigenvectors in the e-basis. For example, if $|a\rangle$ is an eigenvector, express it as

$$|a\rangle = a_1|e_1\rangle + a_2|e_2\rangle + a_3|e_3\rangle$$

where the $a_n = \langle e_n|a\rangle$.

Answer: In this basis, the matrix is

$$\begin{pmatrix} \langle e_1|\hat{A}|e_1\rangle & \langle e_1|\hat{A}|e_2\rangle & \langle e_1|\hat{A}|e_3\rangle \\ \langle e_2|\hat{A}|e_1\rangle & \langle e_2|\hat{A}|e_2\rangle & \langle e_2|\hat{A}|e_3\rangle \\ \langle e_3|\hat{A}|e_1\rangle & \langle e_3|\hat{A}|e_2\rangle & \langle e_3|\hat{A}|e_3\rangle \end{pmatrix} = \begin{pmatrix} 2 & -2i & 2 \\ 2i & 5 & -i \\ 2 & i & 5 \end{pmatrix}$$

The eigenvalues are found by

$$\begin{vmatrix} 2-\lambda & -2i & 2 \\ 2i & 5-\lambda & -i \\ 2 & i & 5-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 0, 6, 6$$

The normalized eigenvector for $\lambda = 0$ is straightforward,

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -i \\ -1 \end{pmatrix} \quad \Rightarrow \quad |a_1\rangle = \frac{1}{\sqrt{6}}\left(2|e_1\rangle - i|e_2\rangle - |e_3\rangle\right)$$

The degenerate $\lambda = 6$ case needs a bit more care, as *all* the equations for the eigenvector components end up being the same. The solutions are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \quad \Rightarrow \quad |a_2\rangle = \frac{1}{\sqrt{2}}\left(|e_2\rangle + i|e_3\rangle\right)$$

$$\frac{1}{\sqrt{3}}\left(\begin{array}{c} 1 \\ i \\ 1 \end{array}\right) \quad \Rightarrow \quad |a_3\rangle = \frac{1}{\sqrt{3}}\left(|e_1\rangle + i|e_2\rangle + |e_3\rangle\right)$$

These are all orthonormal. But in the degenerate subspace, this is not automatic, so if your $\lambda = 6$ solutions did not turn out to be orthogonal, you will need to subtract out a parallel component for one of the solutions.

2. (35 points) Let's find the Fourier transform of a Gaussian in two steps.

(a) Given

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

find

$$\int_{-\infty}^{\infty} dx\, e^{-(x+ia)^2}$$

where a is a real constant. Note that in doing this, you will end up with an integral in the complex plane that is *not* over the real axis! Make a sketch of the contour you use in your integration and show your reasoning explicitly.

Answer: If you change variables to $u = x + ia$, the integral becomes

$$I = \int_{-\infty}^{\infty} dx\, e^{-(x+ia)^2} = \int_{-\infty+ia}^{\infty+ia} du\, e^{-u^2}$$

which is an integral in the complex plane over a line parallel to the x-axis, displaced by ia. We can now complete a rectangular closed contour with the real axis, the contour for I backwards, and the two segments with length a at $x = \pm\infty$:

$$\begin{aligned} \oint_C dz\, e^{-z^2} &= \int_{-\infty}^{\infty} dx\, e^{-x^2} - I + \\ &\quad \lim_{x\to\infty} \int_0^a dy\, e^{-(x+iy)^2} + \lim_{x\to-\infty} \int_a^0 dy\, e^{-(x+iy)^2} \end{aligned}$$

The two length a segments give

$$\lim_{x\to\infty} \int_0^a dy\, e^{-(x+iy)^2} = \lim_{x\to-\infty} \int_a^0 dy\, e^{-(x+iy)^2} = 0$$

Now also notice that e^{-z^2} does not blow up anywhere, and hence is completely analytic, on and within the contour C. Therefore the residue is zero, and

$$\oint_C dz\, e^{-z^2} = \sqrt{\pi} - I + 0 + 0 = 0 \quad \Rightarrow \quad I = \sqrt{\pi}$$

In other words, the ia has no effect.

(b) Now, using (a), find the Fourier transform of a Gaussian $\tilde{f}(k)$, where

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-x^2}$$

Answer: First, notice that

$$x^2 + ikx = \left(x + \tfrac{1}{2}ik\right)^2 + \tfrac{1}{4}k^2$$

Therefore

$$\tilde{f}(k) = \frac{e^{-k^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-(x+ik/2)^2} = \frac{e^{-k^2/4}}{\sqrt{2}}$$

The Fourier transform of a Gaussian is another Gaussian.

3. (30 points)

(a) You have a partial differential equation

$$\frac{\partial^2 f}{\partial \theta^2} + i\frac{\partial f}{\partial t} = 0$$

This has boundary conditions that

(i) $|f(\theta, t)| < \infty$ as $t \to \pm\infty$,

(ii) $f(\theta + n2\pi, t) = f(\theta, t)$ for any integer n.

Write down the general solution for $f(\theta, t)$ satisfying these boundary conditions.

Answer: Look for basis functions of the form $b(\theta, t) = \Theta(\theta)T(t)$. After the usual manipulations, we end up with

$$\frac{\Theta''}{\Theta} + i\frac{\dot{T}}{T} = 0 \quad \Rightarrow \quad -\frac{\Theta''}{\Theta} = i\frac{\dot{T}}{T} = \omega$$

where ω is a complex constant.

The t-equation is

$$\dot{T} + i\omega T = 0 \quad \Rightarrow \quad T = e^{-i\omega t}$$

To satisfy $|T(\theta, t)| < \infty$ as $t \to \pm\infty$, $\text{Im}(\omega) = 0$; in other words, ω must be real.

The θ-equation is

$$\Theta'' + \omega\Theta = 0 \quad \Rightarrow \quad \Theta = e^{\pm im\theta}$$

with $m = \sqrt{\omega}$. To satisfy $\Theta(\theta + n2\pi, t) = \Theta(\theta, t)$ for any integer n, m must be an integer.

The basis functions are

$$b(\theta, t) = A_m e^{im\theta} e^{-im^2 t}, \quad m = 0, \pm 1, \pm 2, \ldots$$

So the general solution is

$$f(\theta, t) = \sum_{m=-\infty}^{\infty} A_m e^{im\theta} e^{-im^2 t}$$

with undetermined constants A_m.

(b) To see the significance of the $+i$ factor in the PDE in (a), look at a close relative:

$$\frac{\partial^2 f}{\partial\theta^2} - i\frac{\partial f}{\partial t} = 0$$

Solve this with the same boundary conditions.

Answer: With the $-i$ factor, everything stays the same, except that the t-solutions have a plus sign in the exponential:

$$b(\theta, t) = A_m e^{im\theta} e^{im^2 t}, \quad m = 0, \pm 1, \pm 2, \ldots$$