Solutions to Final

1. (40 points) Take the projection operator onto a vector $|A\rangle$:

$$\hat{P}_A = \frac{|A\rangle\langle A|}{\langle A|A\rangle}$$

(a) Show that projecting multiple times is equivalent to a single projection: that $\hat{P}_A^2 = \hat{P}_A$.

Answer:

$$\hat{P}_A^2 = \frac{|A\rangle\langle A|}{\langle A|A\rangle}\frac{|A\rangle\langle A|}{\langle A|A\rangle} = \frac{|A\rangle\langle A|A\rangle\langle A|}{\langle A|A\rangle\langle A|A\rangle} = \frac{|A\rangle\langle A|}{\langle A|A\rangle} = \hat{P}_A$$

(b) Say that $|A\rangle$ and all other relevant vectors are part of an N-dimensional vector space spanned by an orthonormal basis set $\{|e_m\rangle\}$, with $m = 1, 2, \ldots, N$ and $\langle e_m|e_n\rangle = \delta_{mn}$. Now show that $\hat{P}_A^\dagger = \hat{P}_A$. *Hint:* The easiest way to do this would be to show that the matrix elements $(\hat{P}_A)_{mn} = (\hat{P}_A)^*_{nm}$.

Answer:

$$(\hat{P}_A)_{mn} = \langle e_m|\hat{P}_A|e_n\rangle = \frac{\langle e_m|A\rangle\langle A|e_n\rangle}{\langle A|A\rangle}$$

$$(\hat{P}_A)^*_{nm} = \frac{\langle e_n|A\rangle^*\langle A|e_m\rangle^*}{\langle A|A\rangle^*} = \frac{\langle e_m|A\rangle\langle A|e_n\rangle}{\langle A|A\rangle}$$

(c) How many linearly independent eigenvectors does $\hat{P}_A$ have? What are these, and what are the eigenvalues? (You may want to revisit this after part (d))

Answer: Since $\hat{P}_A$ is Hermitian, it will have N independent eigenvectors, all corresponding to real eigenvalues. We are looking for vectors $|\psi\rangle$ such that $\hat{P}_A|\psi\rangle = \lambda|\psi\rangle$. In other words,

$$\frac{|A\rangle\langle A|}{\langle A|A\rangle}|\psi\rangle = \frac{\langle A|\psi\rangle}{\langle A|A\rangle}|A\rangle = \lambda|\psi\rangle$$

One solution to this is $|\psi\rangle = |A\rangle$, with eigenvalue $\lambda = 1$. Obviously, projecting any vector parallel to $|A\rangle$ onto $|A\rangle$ will return exactly what we start out with.

But there are $N-1$ more eigenvectors. These are degenerate: they are any set of $N-1$ linearly independent vectors orthogonal to $|A\rangle$. In that case, $\langle A|\psi\rangle = 0$, and $\lambda = 0$ in each case.

(d) Look at the case where $N = 3$, and the column vector consisting of the components $\langle e_m|A\rangle$ is

$$\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

Write down the matrix corresponding to $\hat{P}_A$'s matrix elements arranged into a 3×3 table, in the usual fashion. Then find its eigenvalues and an *orthonormal* set of eigenvectors.

Answer: The inner product is

$$\langle A|A\rangle = \begin{pmatrix} 1 & 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = 1 + 0 + 1 = 2$$

So, using the outer product, the matrix for $\hat{P}_A$ is

$$\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \begin{pmatrix} 1 & 0 & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

The eigenvalues:

$$\begin{vmatrix} \frac{1}{2} - \lambda & 0 & -\frac{i}{2} \\ 0 & -\lambda & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} - \lambda \end{vmatrix} = -\lambda \left[\left(\frac{1}{2} - \lambda \right)^2 - \frac{1}{4} \right] = 0$$

The solutions are $\lambda = 0, 0, 1$, as expected from (c).

$\lambda = 1$ corresponds to the original column vector. Normalizing this, we get

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

For the others, we need to solve

$$\begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a - \frac{i}{2}c \\ 0 \\ \frac{i}{2}a + \frac{1}{2}c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One solution is $c = -ia$, taking $a = 1$, with $b = 0$. Normalizing this, we get

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$$

Finally, we have $b \neq 0$ as a solution. Taking $a = c = 0$, we get

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which is normalized and orthogonal to the other two.

2. (20 points) Find the Fourier transform $\tilde{f}(k)$ of the function

$$f(x) = \frac{1}{x^2 + a^2}$$

where the real number $a > 0$.

Check your answer for $\tilde{f}(k)$: is it real, imaginary, or otherwise? What is its limit as $a \to \infty$? Do your results make sense?

Hint: You will need to treat the $k > 0$ and $k < 0$ cases separately.

Answer: Using the symmetric definition,

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-ikx} \, \frac{1}{x^2 + a^2}$$

This calls for contour integration. Take

$$\frac{1}{\sqrt{2\pi}} \oint_C dz \, \frac{e^{-ikz}}{z^2 + a^2}$$

with C the contour including the infinite semicircle in the *lower* half of the complex plane, where the imaginary part of z is *negative*. The reason is make the exponential go to zero for positive k. (For negative k, we use the upper semicircular contour.)

$$\frac{1}{\sqrt{2\pi}}\oint_C dz\,\frac{e^{-ikz}}{z^2+a^2} = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} dx\,e^{-ikx}\,\frac{1}{x^2+a^2} + \int_0^{-\pi} d\theta\,(ire^{i\theta})\frac{e^{-ikre^{i\theta}}}{r^2e^{2i\theta}+a^2}\right]$$

Now, as $r \to \infty$, $e^{-ikre^{i\theta}} \to 0$ for $k > 0$, so we are left with only our Fourier transform. There is one more complication, however. The contour C is oriented clockwise, so we need an extra minus sign. So

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\oint_C dz\,\frac{e^{-ikz}}{z^2+a^2} = -2\pi i\sum \text{Res}$$

The function integrated has two simple poles, $z = \pm ia$. Of these, only $z = -ia$ is within C. The residue here is

$$\frac{1}{\sqrt{2\pi}}\frac{e^{-ikz}}{(z-ia)(z+ia)}(z+ia)\bigg|_{z=-ia} = \frac{1}{\sqrt{2\pi}}\frac{e^{-ka}}{-2ia}$$

Therefore

$$\tilde{f}(k) = \sqrt{\frac{\pi}{2}}\frac{e^{-ka}}{a} \qquad \text{for } k \geq 0$$

This is for nonnegative k. For $k < 0$, we can just notice that $\tilde{f}(-k) = \tilde{f}(k)^*$, as with every Fourier transform of a real function. We end up with

$$\tilde{f}(k) = \sqrt{\frac{\pi}{2}}\frac{e^{-|k|a}}{a}$$

This is real, as it should be, since $f(x)$ is an even function. As $a \to \infty$, $\tilde{f}(k) \to 0$, which makes sense since then $f(x) \to 0$.

3. (40 points) In doing electrostatics, you will occasionally find yourself solving a PDE that looks like

$$\left[\rho^2\frac{\partial^2}{\partial\rho^2} + 2\rho\frac{\partial}{\partial\rho} - (\mu^2-1)\frac{\partial^2}{\partial\mu^2} - 2\mu\frac{\partial}{\partial\mu}\right]\phi(\rho,\mu) = 0$$

You start looking for product-form solutions $R(\rho)M(\mu)$, subject to the boundary conditions that $|\phi(0,\mu)| < \infty$ and $|\phi(\rho,1)| < \infty$.

(a) Write down the ODE you will have to solve for $M(\mu)$, which includes a constant c that arises from the separation of variables. This ODE will be unfamiliar: you will have to seek a series solution. Find the recursion relationship for the series solution.

Hint: You don't need to try a *generalized* series solution. You will end up with two independent series solutions, one an even function, the other odd, with the same recurrence relationship.

Answer: Putting $R(\rho)M(\mu)$ into the PDE and dividing by $R(\rho)M(\mu)$, we get

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{2\rho}{R}\frac{dR}{d\rho} = \frac{(\mu^2-1)}{M}\frac{d^2M}{d\mu^2} + \frac{2\mu}{M}\frac{dM}{d\mu} = c$$

The ODE for M is then

$$(\mu^2-1)\frac{d^2M}{d\mu^2} + 2\mu\frac{dM}{d\mu} - cM = 0$$

Look for solutions $M = \sum a_n \mu^n$. This gives

$$\sum_{n=0}^{\infty} a_n n(n-1)\mu^n - \sum_{n=0}^{\infty} a_n n(n-1)\mu^{n-2} + \sum_{n=0}^{\infty} 2a_n n\mu^n - \sum_{n=0}^{\infty} c\, a_n\mu^n = 0$$

Shifting the second sum,

$$\sum_{n=0}^{\infty} a_n n(n-1)\mu^n - \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1)\mu^n + \sum_{n=0}^{\infty} 2a_n n\mu^n - \sum_{n=0}^{\infty} c\, a_n\mu^n = 0$$

Note that the $n = -2, -1$ terms give zero, so we really have

$$\sum_{n=0}^{\infty} \left[a_n n(n-1) - a_{n+2}(n+2)(n+1) + 2a_n n - c\, a_n\right]\mu^n = 0$$

This directly results in

$$a_{n+2} = \frac{n(n+1)-c}{(n+2)(n+1)}\, a_n$$

There are two independent series here: One with the even powers of μ, with a nonzero a_0 as an undetermined constant and $a_1 = 0$, and one with the odd powers, with a nonzero a_1 and $a_0 = 0$.

(b) To satisfy the boundary conditions, you will now notice that your series solution for $M(\mu)$ must terminate—you need the solutions to be polynomials. This means that only certain c values will be allowed: those that make the series terminate. Write down an expression for what c values are allowed. Give the first four allowed c values and the corresponding polynomial M's.

Answer: For large n, you see that $a_{n+2} \to a_n$. Such a series will diverge at $\mu = 1$. So we need polynomial solutions. The termination condition is that $c = n(n+1)$ for some value of $n = 0, 1, 2, \ldots$

Let us start with $n = 0$: $c = 0$. This gives $M_0 = a_0$ for the even series; we need to set $a_1 = 0$ to make the odd series go away, since it can't also terminate.

For $n = 1$ we get rid of the even series with $a_0 = 0$, leaving $c = 1(1+1) = 2$. Only a_1 survives, so $M_1 = a_1\mu$.

For $n = 2$ we have an even series again, with $c = 2(2+1) = 6$. We have $a_2 = -3a_0$, so $M_2 = a_0(-3\mu^2 + 1)$.

For $n = 3$ we have an odd series with $c = 3(3+1) = 12$. Then, $a_3 = -\frac{5}{3}$, so $M_3 = a_1(-\frac{5}{3}\mu^3 + \mu)$.

The M_n are known as Legendre polynomials.

(c) Now do the ODE for $R(\rho)$, with your restriction on c values in mind. The solutions should be of a power law form: $R(\rho) = \rho^q$. Find what values q can be, to satisfy the boundary conditions.

Answer: Using $c = n(n+1)$, the R ODE is

$$\rho^2 \frac{d^2R}{d\rho^2} + 2\rho\frac{dR}{d\rho} - n(n+1)R = 0$$

Substituting $R(\rho) = \rho^q$, we get

$$q(q-1) + 2q - n(n+1) = 0 \qquad \Rightarrow \qquad q(q+1) = n(n+1)$$

This has solutions $q = n$ and $q = -n-1$. We throw away the second solution because it blows up at $\rho = 0$, violating the boundary conditions. The solution is

$$R = \rho^n \qquad n = 0, 1, 2, \ldots$$

(d) Put it all together: write down the general solution $\phi(\rho, \mu)$.

Answer: With arbitrary constants A_n,

$$\phi(\rho, \mu) = \sum_{n=0}^{\infty} A_n \rho^n M_n(\mu)$$