Homework Solutions, Chapter 14

2.21 Separate the real and imaginary parts:

$$e^{i(x+iy)} = e^{ix}e^{-y} = e^{-y}(\cos x + i\sin x)$$

Giving $u = e^{-y}\cos x$ and $v = e^{-y}\sin x$. Check the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = -e^{-y}\sin x = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -e^{-y}\cos x = -\frac{\partial v}{\partial x}$$

So the function is analytic.

2.39 To get the series, you can either take successive derivatives or use the binomial theorem. Let's do the binomial:

$$\frac{z}{z^2+9} = \frac{z}{9}\left[1+\frac{z^2}{9}\right]^{-1} = \frac{z}{9}\left[1-\frac{z^2}{9}+\left(\frac{z^2}{9}\right)^2-\ldots\right]$$

The binomial expansion works for $|z^2/9| < 1$, or $|z| < 3$. We can confirm this by looking at the singularities: $z/(z^2+9)$ blows up when $z^2+9=0$, or $z=\pm 3i$. The distance from the origin to $z=\pm 3i$ is, in fact, $|\pm 3i| = 3$. So the series converges for $|z|<3$.

3.6 Path a is along a quarter-circle. Use $z = re^{i\theta}$, with $dz = ire^{i\theta}d\theta$, for constant $r=1$. We get

$$i\int_0^{\pi/2} d\theta\, e^{i2\theta} = \left.\frac{e^{i2\theta}}{2}\right|_0^{\pi/2} = -1$$

Now, path b will give the same result, because we can distort a into b without crossing any non-analytic spots. But let's do this by brute force anyway. We will have two segments, with $z = x+iy$ and $dz = dx + idy$, and $dx = 0$ for constant $x=1$ in the first and $dy=0$ for constant $y=1$ in the second. So we get

$$i\int_0^1 dy\,(1+iy)+\int_1^0 dx\,(x+i) = i\left[y+i\frac{y^2}{2}\right]_0^1+\left[\frac{x^2}{2}+ix\right]_1^0 = i-\frac{1}{2}-\frac{1}{2}-i=-1$$

3.12 Now, $|z|^2$ is non-analytic, so the integrals will be path-dependent. For path a, use $z = x + iy$, where y is constrained to be $y = 2x$. So $z = x(1+2i)$, $|z|^2 = z^* z = 5x^2$, and $dz = dx(1+2i)$, and our integral is:

$$\int_0^1 dx\,(1+2i)5x^2 = (5+10i)\left.\frac{x^3}{3}\right|_0^1 = \frac{5+10i}{3}$$

For path b, we use two segments, with $dz = i dy$ for constant $x = 0$ and $|z|^2 = z^* z = y^2$, and $dz = dx$ for constant $y = 2$ and $|z|^2 = z^* z = x^2 + 4$:

$$i\int_0^2 dy\, y^2 + \int_0^1 dx\,(x^2+4) = i\left.\frac{y^3}{3}\right|_0^2 + \left[\frac{x^3}{3} + 4x\right]_0^1 = \frac{8}{3}i + \frac{1}{3} + 4 = \frac{13+8i}{3}$$

Notice how our results are *not* equal in this case.

7.2 Do it on the unit circle:

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = -i\oint \frac{dz}{z}\frac{1}{5 - 3\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]} = -i\oint \frac{dz}{-\frac{3}{2}z^2 + 5z - \frac{3}{2}}$$

The roots of $-\frac{3}{2}z^2 + 5z - \frac{3}{2} = 0$ are $z_\pm = (-5 \pm \sqrt{25-9})/(-3)$. $z_- = 3$ is outside the unit circle. $z_+ = \frac{1}{3}$ is inside. So

$$I = -i2\pi i\,(\text{Residue at } z_+) = 2\pi \lim_{z\to\frac{1}{3}} \frac{z - \frac{1}{3}}{-\frac{3}{2}z^2 + 5z - \frac{3}{2}} = \frac{2\pi}{-\frac{3}{2}(\frac{1}{3} - 3)} = \frac{\pi}{2}$$

7.15 Use the fact that you're integrating an even function. But first change variables to $s = 2x$; otherwise the usual procedure will not work.

$$I = \int_0^\infty dx\,\frac{\cos 2x}{9x^2+4} = 2\int_0^\infty ds\,\frac{\cos s}{9s^2+16} = \text{Re}\left[\int_{-\infty}^\infty ds\,\frac{e^{is}}{9s^2+16}\right]$$

Do the integral over the infinite half-circle contour:

$$\oint dz\,\frac{e^{iz}}{9z^2+16} = \int_{-\infty}^\infty ds\,\frac{e^{is}}{9s^2+16} + iR\int_0^\pi d\theta\,e^{i\theta}\frac{e^{iRe^{i\theta}}}{9R^2e^{i2\theta}+16}$$

For $0 \le \theta \le \pi$ the last integral is zero, since $\text{Im}[z] \ge 0$, hence $|e^{iz}| \le 1$.

Next we find the roots of $9z^2 + 16 = 0$: $z_\pm = \pm\frac{4}{3}i$, with only z_+ within our contour. So

$$I = \text{Re}\left[2\pi i \lim_{z\to\frac{4}{3}i}\left(z - \frac{4}{3}i\right)\frac{e^{iz}}{9z^2+16}\right] = \text{Re}\left[2\pi i\frac{e^{-\frac{4}{3}}}{9(\frac{4}{3}i + \frac{4}{3}i)}\right] = \frac{\pi e^{-\frac{4}{3}}}{12}$$