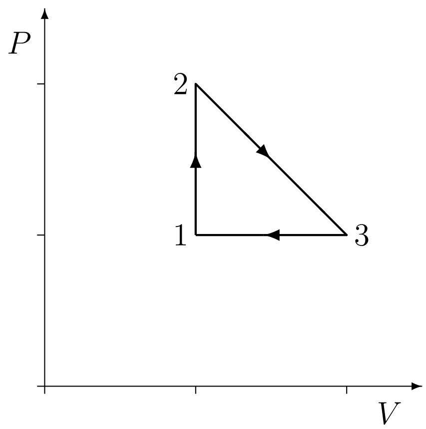
1. (50 points) You have a heat engine that uses a monatomic ideal gas as a working fluid, and that goes through the following cycle, where $P_2 = 2P_1$ and $V_3 = 2V_1$.



(a) Calculate the efficiency of the heat engine. Note, however, that you should not assume that the heat transfer to the engine during $2 \to 3$ is always the same sign. Find out if there are regions on the line segment where $Q < 0$, and if so, account for it as part of Q_h or Q_c accordingly.

Answer: For $1 \to 2$,

$$W_{12} = 0 \quad Q_{12} = \Delta U - W = C_v \Delta T = \frac{3}{2}(P_2V_1 - P_1V_1) = \frac{3}{2}P_1V_1 > 0$$

For $3 \to 1$,

$$\begin{aligned} W_{31} &= -P\Delta V = P_1V_1 \\ Q_{31} &= \Delta U - W = C_p \Delta T = \frac{5}{2}(P_1V_1 - P_1V_3) = -\frac{5}{2}P_1V_1 < 0 \end{aligned}$$

For $2 \to 3$, the equation of the path is

$$P(V) = P_1\left(3 - \frac{V}{V_1}\right)$$

Therefore, since $PV = NkT$, on this path,

$$P_1\left(3 - \frac{V}{V_1}\right)V = NkT \quad \Rightarrow \quad P_1\left(3 - \frac{2V}{V_1}\right)dV = Nk\,dT$$

Now, since $đW = -pdV$ and $đQ = dU - đW$,

$$\begin{aligned} đQ &= \frac{3}{2}Nk\,dT + PdV = P_1\left[\frac{3}{2}\left(3 - \frac{2V}{V_1}\right) + \left(3 - \frac{V}{V_1}\right)\right] dV \\ &= \frac{P_1}{2}\left[15 - 8\frac{V}{V_1}\right] dV \end{aligned}$$

For V increasing from V_1 until $2V_1$, $dV > 0$, and this means that $đQ > 0$ for $V_1 < V < \frac{15}{8}V_1$, and $đQ < 0$ for $\frac{15}{8}V_1 < V < 2V_1$. The two line segments must be accounted for separately, as part of Q_h and as part of Q_c. The positive and negative parts are:

$$Q_+ = \frac{P_1}{2}\int_{V_1}^{\frac{15}{8}V_1} dV\,\left(15 - 8\frac{V}{V_1}\right) = \frac{49}{32}P_1V_1 = 1.53\,P_1V_1$$

$$Q_- = \frac{P_1}{2}\int_{\frac{15}{8}V_1}^{2V_1} dV\,\left(15 - 8\frac{V}{V_1}\right) = -\frac{1}{32}P_1V_1 = -0.03\,P_1V_1$$

The work done from $2 \to 3$ is straightforward; $W_{23} = -\frac{3}{2}P_1V_1$.

Adding positive and negative heats up separately,

$$Q_h = Q_{12} + Q_+ = \frac{97}{32}P_1V_1 = 3.03\,P_1V_1$$

$$Q_c = -Q_{12} - Q_- = \frac{81}{32}P_1V_1 = 2.53\,P_1V_1$$

$$W = -W_{12} - W_{23} - W_{31} = \frac{1}{2}P_1V_1 = 0.50\,P_1V_1$$

The efficiency is

$$e = \frac{W}{Q_h} = \frac{16}{97} = 0.165$$

(b) Find the efficiency of a Carnot engine operating between the maximum and minimum temperatures on this cycle.

Answer: We can start by remembering that isotherms are hyperbolas, and that the hyperbolas move away from the axes as T increases.

Therefore, from symmetry, the maximum temperature on the cycle is midpoint on the $2 \to 3$ part. This is at $P = \frac{3}{2}P_1$ and $V = \frac{3}{2}V_1$, so that

$$T_h = \frac{9P_1V_1}{4Nk}$$

The minimum temperature is at point 1, so $T_c = P_1V_1/Nk$. The maximum efficiency is therefore

$$e_{\rm max} = 1 - \frac{T_c}{T_h} = \frac{5}{9} = 0.556$$

2. (50 points) Take two two-state paramagnet systems A and B, with numbers of particles N_A and N_B, starting at temperatures $T_A > 0$ and $T_B < 0$. Then bring A and B into thermal contact so that they can exchange energy.

(a) Find the equilibrium temperature of the combined system, and find out the direction of energy transfer (A to B or B to A). How are the initial and equilibrium temperatures related—larger, smaller, something more complicated?

Answer: The results for a two-state paramagnet have already been worked out, just use them.

The energy of a two-state paramagnet has the opposite sign of the temperature. Since $T_A > 0$, therefore $U_A < 0$. And since $T_B < 0$, therefore $U_B < 0$. Now look at the temperatures, which depend on the energy densities U/N:

$$\frac{1}{T_A} = \frac{k}{\mu B} \ln\left(\frac{N_A - U_A/\mu B}{N_A + U_A/\mu B}\right) = \frac{k}{\mu B} \ln\left[\frac{1 - U_A/(N_A\mu B)}{1 + U_A/(N_A\mu B)}\right]$$

The combined system is also a two-state paramagnet, and so it will have an equilibrium temperature for $N = N_A + N_B$ and $U = U_A + U_B$:

$$\frac{1}{T_{A+B}} = \frac{k}{\mu B} \ln\left[\frac{1 - (U_A + U_B)/[(N_A + N_B)\mu B]}{1 + (U_A + U_B)/[(N_A + N_B)\mu B]}\right]$$

Now, given what we know about the signs of U_A and U_B, look at the energy densities and the resultant temperatures

$$\frac{U_A}{N_A} < \frac{U_A + U_B}{N_A + N_B} \qquad \Rightarrow \qquad \frac{1}{T_A} > \frac{1}{T_{A+B}}$$

The energy density for paramagnet A *decreases* as it reaches equilibrium. That means the heat flow—the direction of energy transfer—is away from A.

A very similar argument gives

$$\frac{U_B}{N_B} > \frac{U_A + U_B}{N_A + N_B} \qquad \Rightarrow \qquad \frac{1}{T_B} < \frac{1}{T_{A+B}}$$

Therefore, $\frac{1}{T_B}$ increases toward $\frac{1}{T_{A+B}}$ after thermal contact, just like a warmer object coming into contact with a cooler one. But this involves going from a negative initial temperature to even more negative temperatures, and possibly through $T = \mp\infty$ before reaching declining positive temperatures.

Similarly, $\frac{1}{T_A}$ decreases toward $\frac{1}{T_{A+B}}$ like a cold object warming up. This means that T_A increases, but it can possibly go through $T = \pm\infty$ to get to even hotter negative temperatures.

(b) You will see that the rule that heat spontaneously flows from A to B if $T_A > T_B$ is not correct. Find a more correct expression for this rule, and explain why your rule is better by using the concept of entropy.

Answer: From (a), you can see that the behavior of $1/T$ is key. Therefore, heat flows spontaneously from A to B if

$$\frac{1}{T_A} < \frac{1}{T_B}$$

This accounts for negative temperatures as well.

Here's what is happening. $1/T \propto \partial S/\partial U$. You bring two systems with different $1/T$ values together and allow them to exchange energy. What energy one gains must be lost by the other, since the total energy is constant. And the *total* entropy must increase as a result of the energy transfer. Therefore the energy transfer must be from the system with the smaller value of $\partial S/\partial U$—it will lose some entropy, but that will be compensated by the higher entropy gain by the system with a higher $\partial S/\partial U$. That is, after all, what a derivative means.