Homework Solutions 7a (Schroeder Chapter 7)

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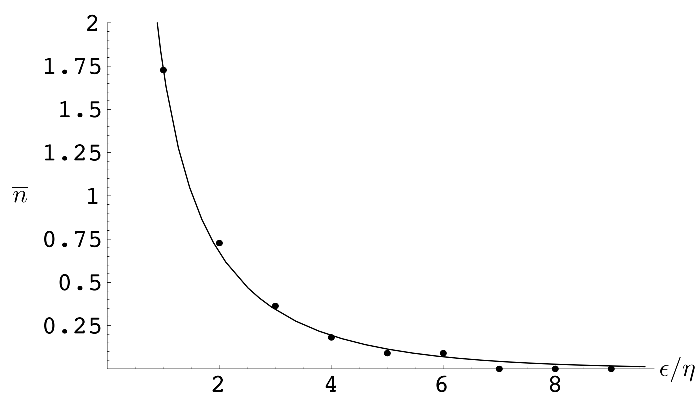
(a) The diagrams for bosons just indicate the occupancy of each level. The ground state is occupied by N particles minus the total number of particles pushed up into excited states. So without this ground state, the diagrams are like

Energy ↑	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
	0	0	0 0	0 0 0	0 0 0 0 0
	0	0	0 0	0 0 0	0 0 0 0 0
	0	0	0 0	0 0 0	0 0 0 0 0
	0	0	0 0	0 0 0	1 0 0 0 0
	0	0	0 0	1 0 0	0 1 0 0 0
	0	0	1 0	0 1 0	0 0 2 1 0
	0	1	0 2	0 1 3	0 1 0 2 4

$q=5$	$q=6$
0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0	0 0 1 1 0 0 0 0 0 0 0
0 0 1 1 0 0 0	0 0 0 0 2 1 1 0 0 0 0
0 0 1 0 2 1 0	0 0 1 0 0 1 0 3 2 1 0
0 1 0 2 1 3 5	0 1 0 2 0 1 3 0 2 4 6

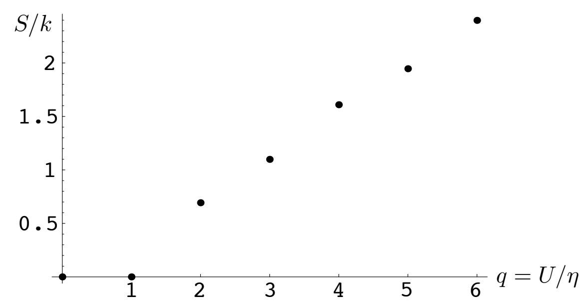
(b) There are 11 possible $q = 6$ states. The average occupancy for the excited states is then the result of adding up the numbers across each row and dividing by 11. The ground state has occupancy $N - \frac{35}{11}$. With the ground state energy set to zero, here is a plot:



(c) At $\epsilon = \mu$, $\bar{n} \to \infty$. If N is large, this means $\mu \approx 0$. The plot above fits the Bose-Einstein distribution for the occupancy graph at $kT = 2.2\eta$. Or, you can say that at $\epsilon = \mu + kT$, the occupancy should be $1/(e-1) \approx 0.6$, and from the data points it looks like this is just above $\epsilon = 2\eta$.

(d) The entropy values turn out to be the same as the fermionic problem, 7.16, I did as an example:

q	Ω	S/k
0	1	0
1	1	0
2	2	0.69
3	3	1.10
4	5	1.61
5	7	1.95
6	11	2.40



And approximating the slope for $1/kT \approx \Delta S/\Delta U$ again gives $kT \approx 2.2\eta$.

24 The neutron star is a bit simpler than the white dwarf. The only particles are neutrons, and $N = M/m_n$. The kinetic energy is then

$$K = \frac{3}{5} N E_F = \frac{3h^2}{40m_n}\left(\frac{M}{m_n}\right)^{5/3}\left(\frac{9}{4\pi^2 R^3}\right)^{2/3} = \frac{\beta}{R^2}$$

with the constants being packed into β. The gravitational potential energy is the same as the white dwarf, so

$$U = K + V = \frac{\beta}{R^2} - \frac{\alpha}{R}$$

with $\alpha = (3/5)GM^2$. Minimizing the energy with $dU/dR = 0$ gives, for a one solar mass neutron star,

$$R = \frac{2\beta}{\alpha} = (0.093)\frac{h^2}{G m_n^{8/3} M^{1/3}} = 1.23 \times 10^4 \text{ m}$$

Its density is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = 2.6 \times 10^{17} \text{ kg/m}^3$$

This is comparable to the density of an atomic nucleus. The Fermi temperature is

$$T_F = \frac{1}{k}\frac{h^2}{8m_n}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8km_n^{5/3}}\left(\frac{9M}{4\pi^2}\right)^{2/3}\frac{1}{R^2} = 6.6\times10^{11}\ \mathrm{K}$$

This is higher than a white dwarf. In any case, the assumption that $T \ll T_F$ is safe.

A neutron star becomes relativistic and unstable when the average neutron kinetic energy is comparable to $m_n c^2 = 940$ MeV. The average $K = \frac{3}{5}kT_F = 34$ MeV for a one solar mass neutron star, which is too small by 1/28. But since $E_F \propto M^{4/3}$, to get into the relativistic regime, we would need $28^{3/4} = 12$ solar masses. This is a crude calculation; astrophysicists calculate the real critical mass at about 2 to 3 solar masses.

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(a) Say $g_{0v} < g_{0c}$. In that case, if μ were to remain constant, there would be fewer electrons added to the conduction band than removed from the valence band. That can't happen, and therefore μ would have to decrease with increasing T. If $g_{0v} > g_{0c}$, μ will have to increase with T.

(b) The number of conduction electrons is

$$N_c = \int_{E_c}^{\infty} dE\, g(E)\bar{n}_{FD}(E) = g_{0c}\int_{E_c}^{\infty} dE\,\frac{\sqrt{E-E_c}}{e^{(E-\mu)/kT}+1}$$

with $kT \ll E_c - \mu$,

$$N_c \approx g_{0c}\int_{E_c}^{\infty} dE\,\frac{\sqrt{E-E_c}}{e^{(E-\mu)/kT}} = \frac{g_{0c}(kT)^{3/2}}{e^{(E_c-\mu)/kT}}\int_0^{\infty} dx\, x^{1/2}e^{-x} = \frac{\sqrt{\pi}g_{0c}(kT)^{3/2}}{2\,e^{(E_c-\mu)/kT}}$$

(c) The occupancy for holes is $1-\bar{n}_{FD}$. Therefore

$$N_h = \int_{-\infty}^{E_v} dE\, g(E)\left[1-\bar{n}_{FD}(E)\right] = g_{0v}\int_{-\infty}^{E_v} dE\,\frac{\sqrt{E_v-E}}{e^{(\mu-E)/kT}+1}$$

This is much like before. When $kT \ll \mu - E_v$, we get

$$N_h \approx g_{0v}\int_{-\infty}^{E_v} dE\,\frac{\sqrt{E_v-E}}{e^{(\mu-E)/kT}} = \frac{g_{0v}(kT)^{3/2}}{e^{(\mu-E_v)/kT}}\int_0^{\infty} dx\, x^{1/2}e^{-x} = \frac{\sqrt{\pi}g_{0v}(kT)^{3/2}}{2\,e^{(\mu-E_v)/kT}}$$

(d) $N_h = N_c$ must be true at any T. Therefore,

$$\frac{g_{0c}}{g_{0v}} = \frac{e^{(E_c-\mu)/kT}}{e^{(\mu-E_v)/kT}} = e^{(E_v+E_c-2\mu)/kT}$$

Taking the log, we find how μ deviates from the center point between the two bands:

$$\mu = \tfrac{1}{2}(E_v + E_c) + \tfrac{1}{2}kT\ln(g_{0v}/g_{0c})$$

(e) For Si, $\ln(0.44/1.09) = -0.91$. Therefore μ shifts by about $\frac{1}{2}\frac{1}{40}(-0.91) = -0.012$ eV. This is considerably smaller than the band gap of 1.1 eV, so the approximations made should work.

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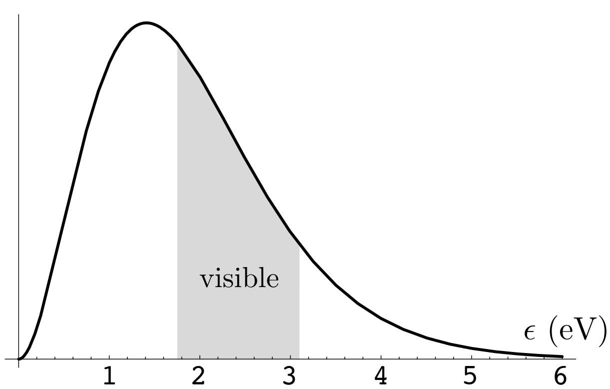
(a) Energy density:

$$\frac{U}{V} = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3} = 0.855\ \text{J/m}^3$$

(b) Spectrum:



(c) Visible light goes from about 1.77 eV to 3.1 eV. With $kT = 0.5$ eV, we have to evaluate

$$\int_{1.77/0.5}^{3.1/0.5} dx\,\frac{x^3}{e^x - 1}$$

and divide this by the full energy in the spectrum,

$$\int_0^\infty dx \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

Done numerically, this results in 0.37.

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(a) To get the number of photons, we need to sum the occupancy of each state. With a factor of 2 to account for the two possible photon spin states,

$$N = 2 \sum_{n_x, n_y, n_z} \frac{1}{e^{hcn/2LkT} - 1}$$

As in the text, approximate this as an integral in spherical coordinates (so the eighth of a sphere gives $\pi/2$), and this gives you the desired result. If you then numerically evaluate the integral, you get

$$N = (2.404)\, 8\pi V \left(\frac{kT}{hc}\right)^3$$

(b) With equation 7.89, we get $\frac{S}{N} = 3.60\, k$

(c) The photon density is

$$\frac{N}{V} = (2.404)\, 8\pi \left(\frac{kT}{hc}\right)^3$$

For 300 K, this gives $5.5 \times 10^{14}/\text{m}^3$. For 1500 K, we get $6.8 \times 10^{16}/\text{m}^3$. And for 2.73 K, we end up with $4.1 \times 10^8/\text{m}^3$. The cosmic microwave photon density in the universe is much larger than the density of matter overall.