1. (40 points) You have N non-interacting spin-$\frac{1}{2}$ particles in a box of volume V. Assume that N is large enough that the single-particle energy levels are well-described by a continuous density of states function $g(E)$.

(a) Write down an expression for the probability density $p(E)$, which gives the probability of a *single particle* to have an energy between E and $E + dE$. Confirm your answer by showing that $p(E) \geq 0$ and that

$$\int_0^\infty dE\, p(E) = 1 \qquad \text{and} \qquad \int_0^\infty dE\, p(E)\, E = \bar{E}$$

Hint: This is a conceptual question; no serious calculation is involved.

Answer: Combining the density of states $g(E)$ with the Fermi-Dirac occupancy $\bar{n}_{FD}(E)$, we have

$$\int_0^\infty dE\, g(E)\bar{n}_{FD}(E) = N \quad \text{and} \quad \int_0^\infty dE\, g(E)\bar{n}_{FD}(E)\, E = U = N\bar{E}$$

So clearly

$$p(E) = \tfrac{1}{N} g(E)\bar{n}_{FD}(E) = \frac{1}{N}\frac{g_0\sqrt{E}}{e^{(E-\mu)/kT}+1}$$

(b) The information entropy per particle is $H = -\sum p_i \ln p_i$. In a continuous case such as ours, this becomes

$$H = -\int dE\, p(E) \ln\left[\epsilon\, p(E)\right]$$

(Here, ϵ is a constant energy scale that appears due to going from the discrete to the continuous limit when defining $g(E)$. It's not important; take it as a constant and otherwise ignore it.)

In this case, we can write the thermodynamic entropy per particle as $S/N = kH$. Use this to calculate the entropy S for our system at absolute zero, $kT = 0$. (This makes sure all your integrals will be doable by hand.)

Answer:

$$S = -kN \int dE\, \frac{g\, \bar{n}_{FD}}{N} \ln\left[\epsilon\, \frac{g\, \bar{n}_{FD}}{N}\right]$$

$$
\begin{aligned}
&= -k\int dE\, g\,\bar{n}_{FD}\left[\ln(g\,\bar{n}_{FD}) + \ln\left(\frac{\epsilon}{N}\right)\right] \\
&= -kN\ln\left(\frac{\epsilon}{N}\right) - k\int dE\, g\,\bar{n}_{FD}\left[\ln(g\,\bar{n}_{FD})\right]
\end{aligned}
$$

Now, let's concentrate on the integral (call it I), using the fact that $\bar{n}_{FD} = 1$ for $0 < E < E_f$ and 0 otherwise.

$$
I = \int dE\, g\,\bar{n}_{FD}\left[\ln(g\,\bar{n}_{FD})\right] = \int_0^{E_F} dE\, g_0\sqrt{E}\ln(g_0\sqrt{E})
$$

Change variables to $u = g_0\sqrt{E}$ and then integrate by parts:

$$
\begin{aligned}
I &= \frac{2}{g_0^2}\int_0^{g_0\sqrt{E_F}} du\, u^2 \ln u = \frac{2u^3}{3g_0^2}\left(\ln u - \tfrac{1}{3}\right)\Bigg|_0^{g_0\sqrt{E_F}} \\
&= \tfrac{2}{3} g_0 E_F^{3/2}\left(\ln(g_0\sqrt{E_F}) - \tfrac{1}{3}\right)
\end{aligned}
$$

We now use the results, derived in the books, for

$$
E_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} \quad \text{and} \quad g_0 = \frac{\pi V (8m)^{3/2}}{2h^3}
$$

Lots of cancellations occur, and we get

$$
I = N\left(\ln(g_0\sqrt{E_F}) - \tfrac{1}{3}\right)
$$

Putting everything back together for the entropy, and joining everything in one logarithm,

$$
S = kN\ln\left(\frac{N\,e^{1/3}}{\epsilon\, g_0\sqrt{E_F}}\right) = kN\ln\left(\frac{2e^{1/3}}{3\epsilon}E_F\right) = kN\ln\left[\frac{e^{1/3}h^2}{12m\epsilon}\left(\frac{3N}{\pi V}\right)^{2/3}\right]
$$

2. (40 points) You have a gas of noninteracting spin-0 particles with a density of states given by

$$
g(E) = g_0\,\delta(E) + g_e\, H(E-\Delta)
$$

where δ is the Dirac delta function and H is the unit step function, such that $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$. Δ, g_0, and g_e are constants. (This is a highly artificial $g(E)$, intended to emphasize a separate ground state and a continuum of excited states lying an energy gap Δ above, and to keep all your integrals easy.)

(a) Find expressions for the population of the ground state, N_0, and the population of the excited states, N_e, at any temperature T. (These will not necessarily look pretty.)

Answer: We need to use

$$N = \int_{-\infty}^{\infty} dE\, g(E)\bar{n}_{BE}(E) = g_0 \frac{1}{e^{-\mu/kT} - 1} + g_e \int_{\Delta}^{\infty} dE\, \frac{1}{e^{(E-\mu)/kT} - 1}$$

The first term is N_0, the second N_e. To do the integral, change variables to $u = e^{(E-\mu)/kT} - 1$, resulting in

$$N_e = -g_e\, kT\, \ln\left(1 - e^{(\mu-\Delta)/kT}\right)$$

(b) Find the ratio N_e/N_0 at the limits where kT is very small and very large. You may need to make assumptions about how the chemical potential μ behaves as a function of T (increases/decreases faster/slower than kT and so forth). Make your assumptions explicit, and provide physical explanations for the behavior of both N_e/N_0 and μ.

Answer: At $T = 0$, $N = N_0$ and $N_e = 0$. Therefore $\mu = 0$ at $T = 0$. For small but finite kT, we can make approximations. μ will be very small and negative, and provided $|\mu|$ grows slower than kT,

$$N_0 \approx g_0 \frac{1}{1 - \mu/kT - 1} = -g_0 \frac{kT}{\mu}$$

$$N_e \approx g_e\, kT\, e^{(\mu-\Delta)/kT}$$

Therefore

$$\frac{N_e}{N_0} \approx \frac{g_e}{g_0} |\mu|\, e^{(\mu-\Delta)/kT} \ll 1$$

For large T, if $|\mu|$ still grows slower than kT so that $|\mu| \ll kT$, the approximation for N_0 remains the same. But now,

$$N_e \approx -g_e\, kT\, \ln\left(1 - 1 - \frac{\mu - \Delta}{kT}\right) = g_e\, kT\, \ln\left|\frac{kT}{\Delta - \mu}\right|$$

In this case,

$$\frac{N_e}{N_0} \approx \frac{g_e}{g_0} |\mu| \ln \left| \frac{kT}{\Delta - \mu} \right|$$

which increases with T, as it should. After all, more particles are now occupying excited states at the expense of the ground state population.

3. (20 points) In the Ising model, no matter what the dimensions, there is a symmetry between up and down spin orientations.

(a) Write down the expressions for the partition function and the probabilities of spin configurations, and show that the expectation value for the spin orientation is $\bar{s} = 0$ at all temperatures.

Answer: Because of the symmetry between spin orientations, each spin configuration will have another opposite spin configuration where all the spins have been reversed, but the energy will be exactly the same because the number of aligned and mis-aligned spins will not change. Therefore, if c stands for a particular configuration,

$$Z = \sum_c e^{-\beta U_c} \qquad \text{and} \qquad P(c) = \frac{1}{Z}$$

Call the configuration where all the spins have been reversed $\tilde{c}$. Clearly $P(\tilde{c}) = P(c)$. Say $\bar{s}_c$ is the average spin for any particular configuration c (as opposed to the overall average among all configurations $\bar{s}$). It's obvious that $\bar{s}_{\tilde{c}} = -\bar{s}_c$, since all the spins will have been flipped. Therefore,

$$\bar{s} = \sum_c P(c)\bar{s}_c = 0$$

because for each c, there is an equally probable $\tilde{c}$ included in the sum that gives an exactly opposite contribution, canceling out $\bar{s}_c$.

(b) How do you reconcile this with the fact that at low enough temperatures, the Ising model exhibits a non-zero magnetization?

Answer: Yes, $\bar{s} = 0$ when averaged over all configurations, but in the particular configuration you end up in, $\bar{s}_c \neq 0$ is perfectly possible. All $\bar{s} = 0$ means is that $\bar{s}_{\tilde{c}} = -\bar{s}_c$ is equally likely.

(c) In the mean-field approximation to the Ising model, we find $\bar{s} \neq 0$ is possible for $T < T_c$. Again, how do you reconcile this with the finding that $\bar{s} = 0$ at all temperatures? [*Hint:* The $\bar{s}$ symbol in the mean-field approximation does not exactly mean the same thing as $\bar{s}$ as a general probabilistic mean as in part (a). Explaining what the difference is will help you answer this question.]

Answer: The $\bar{s}$ symbol in the book's mean field theory is actually what I've been calling $\bar{s}_c$ here. Note that with spontaneous symmetry breaking, the two stable $\bar{s}_c \neq 0$ configurations have equal and opposite $\bar{s}_c$ values, and are equally probable—averaging out to zero.