1. (50 points) You have a spin-$\frac{1}{2}$ system in $|\psi\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}\, e^{i\phi}|-\rangle$.

(a) By repeated measurements of particles prepared in the same state $|\psi\rangle$, you determine that $\mathcal{P}_{S_x=\hbar/2} = p_x$, $\mathcal{P}_{S_y=\hbar/2} = p_y$, and $\mathcal{P}_{S_z=\hbar/2} = p_z$. Show that if you get the same set of probabilities p_x, p_y, p_z for another state $|\varphi\rangle$, then $|\varphi\rangle = e^{i\xi}|\psi\rangle$, where ξ is a real constant. In other words, show that knowing p_x, p_y, p_z determines θ and ϕ uniquely, except for a physically inconsequential overall phase for the state.

Remember: Square roots and inverse trigonometric functions are multiple-valued functions!

Answer: The probabilities are

$$p_x = \left|{}_x\langle+|\psi\rangle\right|^2 = \tfrac{1}{2}\left(1 + \sin\theta\cos\phi\right)$$

$$p_y = \left|{}_y\langle+|\psi\rangle\right|^2 = \tfrac{1}{2}\left(1 + \sin\theta\sin\phi\right)$$

$$p_z = \left|\langle+|\psi\rangle\right|^2 = \tfrac{1}{2}\left(1 + \cos\theta\right)$$

We can take $0 \le \theta \le \frac{\pi}{2}$, such that $\sin\theta \ge 0$ and $\cos\theta \ge 0$. This is because if we take states with other θ angles that give the same p_z, either the state differs from $|\psi\rangle$ in its overall phase, which is unimportant, or it involves a relative phase that can be absorbed into ϕ. Given that, to also get p_x and p_y the same, $\sin\phi$ and $\cos\phi$ must also be the same, which is only possible with the same $0 \le \phi < 2\pi$.

(b) You prepare particles in state $|\psi\rangle$ by measuring an observable O, where $O|\psi\rangle = a|\psi\rangle$, and where the eigenvalues of O are a, b, with $a \ne b$. You then use a particle only after a measurement resulting in $O = a$. Find the matrix representation of O in the $\{|+\rangle, |-\rangle\}$ basis.

Answer: The eigenvectors of a nondegenerate Hermitian operator such as O are automatically orthogonal. Since this is a 2×2 vector space, there is just one normalized vector $|\varphi\rangle$ orthogonal to $|\psi\rangle$ up to a phase factor. This is easy to construct:

$$|\varphi\rangle = \sin\tfrac{\theta}{2}|+\rangle - \cos\tfrac{\theta}{2}\, e^{i\phi}|-\rangle$$

Therefore $O|\psi\rangle = a|\psi\rangle$, $O|\varphi\rangle = b|\varphi\rangle$, and

$$\langle\psi|O|\psi\rangle = a, \quad \langle\varphi|O|\varphi\rangle = b, \quad \langle\psi|O|\varphi\rangle = \langle\varphi|O|\psi\rangle = 0$$

We now need matrix elements such as

$$\langle +|O|+\rangle = \langle +|IOI|+\rangle \quad \text{with} \quad I = |\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|$$

So

$$\begin{aligned}\langle +|O|+\rangle &= \langle +|\left(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|\right) O \left(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|\right)|+\rangle \\ &= \langle +|\psi\rangle a \langle\psi|+\rangle + \langle +|\varphi\rangle b \langle\varphi|+\rangle\end{aligned}$$

and similarly with the other matrix elements. Putting everything together,

$$O \doteq \begin{pmatrix} a\cos^2\frac{\theta}{2} + b\sin^2\frac{\theta}{2} & \frac{1}{2}(a-b)\sin\theta\, e^{-i\phi} \\ \frac{1}{2}(a-b)\sin\theta\, e^{i\phi} & b\cos^2\frac{\theta}{2} + a\sin^2\frac{\theta}{2} \end{pmatrix}$$

2. (50 points) You have a spin-1 particle which, at time $t = 0$, you measure to have $S_n = +\hbar$ along an axis with $\theta = \frac{\pi}{3}$ and $\phi = \frac{\pi}{2}$. The particle's state then evolves in time in a manner determined by a Hamiltonian H. In the basis in which S_z is diagonal, all the matrix elements of this Hamiltonian are zero, except for

$$\langle 0|H|0\rangle = \epsilon \qquad \text{and} \qquad \langle -1|H|1\rangle = \langle 1|H|-1\rangle^* = i\epsilon$$

(a) What are the possible energy values you will measure, and with what probability, at time t?

Answer: The spin operator is

$$S_n = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta = \tfrac{1}{2}\left(\sqrt{3}S_y + S_z\right)$$

Therefore

$$S_n \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & -i\sqrt{\frac{3}{2}} & 0 \\ i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} \\ 0 & i\sqrt{\frac{3}{2}} & -1 \end{pmatrix}$$

We are looking for the eigenstate with eigenvalue $+\hbar$; solve for

$$\frac{\hbar}{2}\begin{pmatrix} 1 & -i\sqrt{\frac{3}{2}} & 0 \\ i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} \\ 0 & i\sqrt{\frac{3}{2}} & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The normalized solution is

$$\frac{1}{\sqrt{|a|^2+|b|^2+|c|^2}}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3i \\ \sqrt{6} \\ i \end{pmatrix}$$

This is the initial state, at $t = 0$, in the basis where S_z is diagonal:

$$|\psi(0)\rangle = \tfrac{1}{4}\left(-3i|1\rangle + \sqrt{6}|0\rangle + i|-1\rangle\right)$$

Now let's diagonalize H.

$$H \doteq \begin{pmatrix} 0 & 0 & -i\epsilon \\ 0 & \epsilon & 0 \\ i\epsilon & 0 & 0 \end{pmatrix}$$

The eigenvalues are the observable energies:

$$\begin{vmatrix} -E & 0 & -i\epsilon \\ 0 & \epsilon - E & 0 \\ i\epsilon & 0 & -E \end{vmatrix} = 0 \qquad \Rightarrow \qquad E_1 = -\epsilon, E_2 = \epsilon, E_3 = \epsilon$$

Note the degeneracy—we have to be careful with it.

The eigenvectors are, after some algebra:

$$|E_1\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} \qquad |E_2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |E_3\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

Note that we have to make sure that $\langle E_2|E_3\rangle = 0$, as this will not be automatic.

Now we expand $|\psi\rangle$ in the energy basis:

$$\begin{aligned} |\psi(0)\rangle &= \sum_n |E_n\rangle\langle E_n|\psi(0)\rangle \\ &= \frac{1}{4\sqrt{2}}\left[(-1-3i)|E_1\rangle + 2\sqrt{3}|E_2\rangle + (1-3i)|E_3\rangle\right] \end{aligned}$$

Therefore

$$|\psi(t)\rangle = \frac{1}{4\sqrt{2}}\left[(-1-3i)e^{i\frac{\epsilon}{\hbar}t}|E_1\rangle + 2\sqrt{3}e^{-i\frac{\epsilon}{\hbar}t}|E_2\rangle + (1-3i)e^{-i\frac{\epsilon}{\hbar}t}|E_3\rangle\right]$$

Reading off the probabilities,

$$\mathcal{P}_{E=-\epsilon} = \tfrac{5}{16} \qquad \mathcal{P}_{E=\epsilon} = \tfrac{11}{16}$$

(b) Find $\mathcal{P}_{S_x=\hbar}$, $\mathcal{P}_{S_x=0}$, and $\mathcal{P}_{S_x=-\hbar}$ at time t.

Answer: We need the eigenstates of

$$S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

These are

$$|1\rangle_x \doteq \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad |0\rangle_x \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad |-1\rangle_x \doteq \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

We also have

$$|\psi(t)\rangle \doteq \frac{1}{8} \begin{pmatrix} (-1-3i)e^{i\frac{\epsilon}{\hbar}t} + (1-3i)e^{-i\frac{\epsilon}{\hbar}t} \\ 2\sqrt{6}\, e^{-i\frac{\epsilon}{\hbar}t} \\ (-3+i)e^{i\frac{\epsilon}{\hbar}t} + (3+i)e^{-i\frac{\epsilon}{\hbar}t} \end{pmatrix}$$

Therefore

$$\mathcal{P}_{S_x=\hbar} = |{}_x\langle 1|\psi(t)\rangle|^2 = 0.5603 + 0.3882\cos(2\epsilon t/\hbar + 3.786)$$

$$\mathcal{P}_{S_x=0} = |{}_x\langle 0|\psi(t)\rangle|^2 = 0.3125 + 0.3125\cos(2\epsilon t/\hbar - 5.356)$$

$$\mathcal{P}_{S_x=-\hbar} = |{}_x\langle -1|\psi(t)\rangle|^2 = 0.1272 + 0.1239\cos(2\epsilon t/\hbar - 0.136)$$

These values add up to 1, so they look trustworthy.