QM Exam 3 (or 4) Solutions

1. (50 points) Page 303 of McIntyre describes "coherent states" of a simple harmonic oscillator. You will investigate these states.

(a) Construct a coherent state $|\kappa\rangle$ by

$$|\kappa\rangle = N e^{\kappa a^\dagger}|0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Find the normalization N and the coefficients c_n. Here κ is an arbitrary complex number, with $\kappa = re^{i\theta}$ where r and θ are κ's magnitude and phase.

Answer:

$$N e^{\kappa a^\dagger}|0\rangle = N \sum_{n=0}^{\infty} \frac{1}{n!} \kappa^n a^{\dagger\, n}|0\rangle = N \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \kappa^n |n\rangle \quad \Rightarrow \quad c_n = N \frac{\kappa^n}{\sqrt{n!}}$$

Normalizing:

$$\langle \kappa | \kappa \rangle = |N|^2 \sum_{n=0}^{\infty} \frac{1}{n!} (\kappa^* \kappa)^n = |N|^2 e^{|\kappa|^2} = 1 \quad \Rightarrow \quad |N| = e^{-r^2/2}$$

(b) Show that a coherent state $|\kappa\rangle$ is an eigenstate of the lowering operator a. Find the eigenvalue.

Answer: With an appropriate shift of indices, $m = n - 1$, we get

$$\begin{aligned} a|\kappa\rangle &= N \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \kappa^n \sqrt{n} |n-1\rangle = \kappa\, N \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n-1)!}} \kappa^{n-1} |n-1\rangle \\ &= \kappa\, N \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \kappa^m |m\rangle = \kappa |\kappa\rangle \end{aligned}$$

In other words, the eigenvalue is κ.

(c) Show that coherent states always remain coherent states. In other words, show that $U(t)|\kappa\rangle = |\kappa(t)\rangle$, where $|\kappa(t)\rangle$ is also a coherent state, but the number κ labeling the state changes over time. Find $\kappa(t)$. *Hint:* Remember that an overall phase does not matter with quantum states.

Answer: Since the $|n\rangle$ are H eigenstates,

$$\begin{aligned}|\kappa(t)\rangle &= e^{-i\frac{t}{\hbar}H}|\kappa\rangle = N\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\kappa^n e^{-i\omega\left(n+\frac{1}{2}\right)t}|n\rangle \\ &= e^{-i\omega t/2}N\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\kappa e^{-i\omega t}\right)^n|n\rangle\end{aligned}$$

The phase up front does not matter; we can absorb it into N. So we end up with a coherent state remaining coherent over time, with

$$\kappa(t) = \kappa e^{-i\omega t} \qquad \text{and} \qquad N = e^{-\left(|\kappa(t)|^2 - i\omega t\right)/2}$$

(Whether you adjust N or not is unimportant, since the overall phase does not matter. It just looks better this way.)

(d) Using the previous result, find the time-dependent expectation values $\langle x\rangle = \langle\kappa(t)|x|\kappa(t)\rangle$ and $\langle p\rangle = \langle\kappa(t)|p|\kappa(t)\rangle$. Your answers for $\langle x\rangle$ and $\langle p\rangle$ should be sines or cosines; use the magnitude and phase r and θ where $\kappa(0) = re^{i\theta}$. *Hint:* $\langle\psi|a^\dagger = (a|\psi\rangle)^\dagger$ for any state $|\psi\rangle$.

Answer:

$$\begin{aligned}\langle\kappa(t)|x|\kappa(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\kappa(t)|a^\dagger|\kappa(t)\rangle + \langle\kappa(t)|a|\kappa(t)\rangle\right] \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\kappa(t)|\kappa(t)^*|\kappa(t)\rangle + \langle\kappa(t)|\kappa(t)|\kappa(t)\rangle\right] \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left[\kappa(t)^* + \kappa(t)\right] = \sqrt{\frac{\hbar}{2m\omega}}r\left[e^{-i\theta}e^{i\omega t} + e^{i\theta}e^{-i\omega t}\right] \\ &= \sqrt{\frac{2\hbar}{m\omega}}\,r\cos(\omega t - \theta)\end{aligned}$$

$$\langle\kappa(t)|p|\kappa(t)\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle\kappa(t)|a^\dagger|\kappa(t)\rangle - \langle\kappa(t)|a|\kappa(t)\rangle\right]$$

$$
\begin{aligned}
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle\kappa(t)|\kappa(t)^*|\kappa(t)\rangle - \langle\kappa(t)|\kappa(t)|\kappa(t)\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\kappa(t)^* - \kappa(t)\right] = i\sqrt{\frac{\hbar m\omega}{2}} r\left[e^{-i\theta}e^{i\omega t} - e^{i\theta}e^{-i\omega t}\right] \\
&= -\sqrt{2\hbar m\omega}\, r\sin(\omega t - \theta)
\end{aligned}
$$

Note that $\langle p\rangle = m\frac{d}{dt}\langle x\rangle$.

(e) There is an identity for operators A and B:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

provided that $[A,[A,B]] = [B,[A,B]] = 0$. Use this identity to find

$$\langle e^{\lambda x}\rangle = \langle\kappa(t)|e^{\lambda\sqrt{\hbar/2m\omega}\left(a^\dagger + a\right)}|\kappa(t)\rangle$$

Here, λ is an arbitrary constant.

Answer: Now, $[a,a^\dagger] = 1$, and therefore $[a,[a,a^\dagger]] = [a^\dagger,[a,a^\dagger]] = 0$. Since $|\kappa(t)\rangle$ is an eigenstate of a with eigenvalue $\kappa(t)$, it's also an eigenstate of a function of a, with eigenvalue the function applied to $\kappa(t)$.

$$
\begin{aligned}
\langle e^{\lambda x}\rangle &= \langle\kappa(t)|e^{\lambda\sqrt{\hbar/2m\omega}\,a^\dagger}e^{\lambda\sqrt{\hbar/2m\omega}\,a}e^{-\frac{1}{2}\lambda^2\frac{\hbar}{2m\omega}\left[a^\dagger,a\right]}|\kappa(t)\rangle \\
&= e^{\frac{\lambda^2\hbar}{4m\omega}}\langle\kappa(t)|e^{\lambda\sqrt{\hbar/2m\omega}\,a^\dagger}e^{\lambda\sqrt{\hbar/2m\omega}\,a}|\kappa(t)\rangle \\
&= e^{\frac{\lambda^2\hbar}{4m\omega}}e^{\lambda\sqrt{\hbar/2m\omega}\,\kappa(t)^*}e^{\lambda\sqrt{\hbar/2m\omega}\,\kappa(t)} \\
&= e^{\frac{\lambda^2\hbar}{4m\omega}}e^{\lambda\sqrt{\hbar/2m\omega}[\kappa(t)^*+\kappa(t)]} = e^{\frac{\lambda^2\hbar}{4m\omega}}e^{\lambda\langle x\rangle}
\end{aligned}
$$

(f) McIntyre tells you that coherent states have Gaussian wave functions that don't change their shape over time. Now you can prove this. The position and momentum probability distributions are Gaussians and retain their shape over time, even though the expectation values change. This means that the central moments of the distributions, $\langle\kappa(t)|\left(x-\langle x\rangle\right)^n|\kappa(t)\rangle$ and $\langle\kappa(t)|\left(p-\langle p\rangle\right)^n|\kappa(t)\rangle$, do not depend on time, and are appropriate for Gaussian distributions, for all $n = 0, 1, 2 \ldots$ Show this.

Hint: Start with writing $\langle e^{\lambda(x-\langle x\rangle)}\rangle$ using both (e) and the series expansion for an exponential. This will allow you to simply read off the moments; do a virtually identical calculation for p rather than x. To check that the moments are Gaussian, use

$$\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-(x-\mu)^2/a^2}(x-\mu)^{2n} = \frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n}$$

$$\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-(x-\mu)^2/a^2}(x-\mu)^{2n+1} = 0$$

where $n = 0, 1, 2, \ldots$

Answer: Since $\langle x\rangle$ is an ordinary number, not an operator,

$$\langle e^{\lambda(x-\langle x\rangle)}\rangle = e^{-\lambda\langle x\rangle}\langle e^{\lambda x}\rangle = e^{\frac{\lambda^2\hbar}{4m\omega}}$$

Now, equating the expansions of the exponentials, we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle (x-\langle x\rangle)^n\right\rangle \lambda^n = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\hbar}{4m\omega}\right)^m \lambda^{2m}$$

Since this must be true for *any* λ, this can only happen if the factors multiplying each power of λ are always the same on each side of the equation. We see that there are no odd powers of λ on the right side. Therefore, when $n = 2m + 1$, we have, for any $m = 0, 1, 2, \ldots$

$$\left\langle (x-\langle x\rangle)^{2m+1}\right\rangle = 0$$

So, just as it should be for a Gaussian, the odd central moments are zero. For the even moments, we look at $n = 2m$, getting

$$\left\langle (x-\langle x\rangle)^{2m}\right\rangle = \frac{(2m)!}{m!}\left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^{2m}$$

These are the moments for a Gaussian distribution with $a = \sqrt{\hbar/m\omega}$.

We can do the same calculation with momentum:

$$\langle e^{\lambda(p-\langle p\rangle)}\rangle = e^{-\lambda\langle p\rangle}\langle e^{\lambda p}\rangle = e^{\frac{\lambda^2\hbar m\omega}{4}}$$

Equating the expansions of the exponentials,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle (p - \langle p \rangle)^n \rangle \lambda^n = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\hbar m \omega}{4} \right)^m \lambda^{2m}$$

Therefore, when $n = 2m + 1$,

$$\left\langle (p - \langle p \rangle)^{2m+1} \right\rangle = 0$$

When $n = 2m$,

$$\left\langle (p - \langle p \rangle)^{2m} \right\rangle = \frac{(2m)!}{m!} \left(\frac{1}{2} \sqrt{\hbar m \omega} \right)^{2m}$$

These are the moments for a Gaussian distribution with $a = \sqrt{\hbar m \omega}$.

2. (20 points) You have a Hamiltonian $H = H_0 + H'$, with

$$H_0 \doteq \frac{p^2}{2\mu} - \frac{e^2}{4\pi\epsilon_0 r} \qquad H' \doteq \tfrac{1}{2}\mu\omega^2 r^2$$

in spherical coordinates. The frequency ω is a small parameter.

(a) Find $E^{(1)}_{nlm}$, the first-order corrections to the eigenvalues of H_0 due to the perturbation H'. *Hint:* You don't need to do any integrals; just use a result provided in your textbook.

Answer: The unperturbed Hamiltonian H_0 is for a Hydrogen atom. The perturbation H' is a simple harmonic oscillator. So the unperturbed states are Hydrogen states.

We will need the diagonal matrix elements of H',

$$\begin{aligned} \langle nlm | H' | nlm \rangle &= \tfrac{1}{2}\mu\omega^2 \langle nlm | r^2 | nlm \rangle = \tfrac{1}{2}\mu\omega^2 \langle r^2 \rangle \\ &= \tfrac{1}{4}\mu\omega^2 a_0^2 n^2 \left[5n^2 + 1 - 3l(l+1) \right] \end{aligned}$$

using equation 8.89 in the textbook for $\langle r^2 \rangle$.

But now, one concern is that Hydrogen energy levels are degenerate: $E^{(0)}_{nlm}$ depends on n but not on l and m. So we have degenerate subspaces for each n, and we need to diagonalize the perturbation in these subspaces. So we have to look at the off-diagonal matrix elements

$$\begin{aligned} \langle nlm|H'|n'l'm'\rangle &= \tfrac{1}{2}\mu\omega^2\langle nlm|r^2|n'l'm'\rangle \\ &= \tfrac{1}{2}\mu\omega^2\delta_{ll'}\delta_{mm'}\int_0^\infty dr\, r^4 R_{nl}(r)R_{n'l}(r) \end{aligned}$$

The radial integral might not be zero for some $n \neq n'$, so there may be some nonzero off-diagonal matrix elements. But since the perturbation is a central force, it has no angular dependence—the orthonormality of the Y_l^m's give the $\delta_{ll'}\delta_{mm'}$ factors above. This means that H' is diagonal *within every degenerate subspace!*

Therefore there is no diagonalization to be done, and

$$E^{(1)}_{nlm} = \tfrac{1}{4}\mu\omega^2 a_0^2 n^2\left[5n^2 + 1 - 3l(l+1)\right]$$

(b) Do you expect these energy corrections to be reasonably accurate for all n? Only for even n, odd n, large n, small n, prime n every other Tuesday—when is the perturbation approximation valid and when does it break down? Provide a physical argument.

Answer: You can trust these corrections only for small n. You notice that the corrections go like n^4, so that even though the simple harmonic oscillator energy scale $\hbar\omega$ (or the correction energy scale $\mu\omega^2 a_0^2$) may be much smaller than typical Hydrogen atom energies, eventually n^4 becomes large enough to violate the assumption that the simple harmonic oscillator part of the Hamiltonian is a small perturbation.

Physically, you can also think of this in terms of the radius r. $H' \ll H_0$ only for small enough r. But as n gets large, the Hydrogen $\langle r\rangle$ becomes large, since in equation 8.89, $\langle r\rangle$ grows like n^2. So for large n, the electron will more often be in a region where $H' \gg H_0$, so that then, the Coulomb potential is a perturbation to the simple harmonic oscillator!

3. (30 points) Say we have a gas of noninteracting identical spin-$\frac{1}{2}$ particles in two dimensions. For each individual particle, the Hamiltonian is

$$H = \frac{1}{2m}\left[(p_x - \alpha y)^2 + p_y^2\right]$$

where α is a constant.

(a) Find the eigenstates and eigenvalues of this Hamiltonian. Here are some hints:

- Try product-form solutions that look like $\psi = f(y)\, e^{ikx}$.
- A polynomial $ay^2 + by + c$ can be rewritten as $a(y - y_0)^2 + E_0$, with constants y_0 and E_0.
- You shouldn't have to solve any differential equations: the 1D Hamiltonian you end up with should look like a Hamiltonian we have already solved.

Answer: Plugging in $\psi = f(y)\, e^{ikx}$, and using $p_x = -i\hbar\partial/\partial x$ etc.,

$$H\psi = \frac{1}{2m}\left[\left(-i\hbar\frac{\partial}{\partial x} - \alpha y\right)^2 - \hbar^2\frac{\partial^2}{\partial y^2}{}^2\right] f(y)\, e^{ikx} = E f(y)\, e^{ikx}$$

After performing the derivatives and canceling out the e^{ikx} factors, you end up with

$$-\frac{\hbar^2}{2m}\frac{d^2 f}{dy^2} + \frac{1}{2m}\left[\hbar^2 k^2 - 2\hbar\alpha k y + \alpha^2 y^2\right] f = Ef$$

The polynomial in y can be rewritten as $(\alpha y - \hbar k)^2$. Therefore, if we change variables to $y' = y - \frac{\hbar}{\alpha}k$ with $dy' = dy$, we get

$$-\frac{\hbar^2}{2m}\frac{d^2 f}{dy'^2} + \frac{\alpha^2}{2m}y'^2 f = Ef$$

This is a simple harmonic oscillator equation with $\omega = \alpha/m$. Therefore we can just use the known solutions:

$$\psi_{nk} = \varphi_n\left(y - \frac{\hbar}{\alpha}k\right) e^{ikx} \qquad E_n = \frac{\hbar\alpha}{m}\left(n + \tfrac{1}{2}\right)$$

where $n = 0, 1, 2, \ldots$; $-\infty < k < \infty$; and φ_n are the known simple harmonic oscillator eigenstates.

(b) Can more than two of our identical spin-$\frac{1}{2}$ particles in this 2D gas have the same energy? (The factor of 2 comes from accounting for opposite spin particles going into the same spatial state.)

Answer: Yes. No more than two of our fermions can go into the same state, but if the energies are degenerate, more than two can have the same energy. In this case, E depends on n only, not on k, so we can pile as many fermions as we like into the same E_n levels as long as they have different k's.

Note: This is actually a problem about "Landau levels"—they come about when you have charged particles in a constant external magnetic field. The Hamiltonian in that case does not have a $V(\vec{r})$ term, as the magnetic field comes from a vector potential $\vec{A}$. You account for it by shifting the momentum $\vec{p}$. Look it up!