Homework Solutions 2 (Ryden Chapter 3)

3 On a circle, r is constant, so $dr = 0$. Therefore $dl = R\sin(r/R)\,d\theta$, and the circumference is

$$C = R\sin\left(\frac{r}{R}\right)\int_0^{2\pi} d\theta = 2\pi R\sin\left(\frac{r}{R}\right)$$

The deviation from flatness is

$$\Delta C = 2\pi r - 2\pi R\sin\left(\frac{r}{R}\right) \approx 2\pi r - 2\pi R\left(\frac{r}{R} - \frac{1}{3!}\frac{r^3}{R^3}\right) = \frac{\pi}{3}\frac{r^3}{R^2}$$

where the approximate value is for $r \ll R$. To get $\Delta C > 1$ m, we need $r > 0.0053R = 3.4 \times 10^4$ m.

4 With $\kappa = +1$, a sphere, the largest triangle is a great circle, with area half the sphere: $2\pi R^2$.

With $\kappa = 0$, a plane, there is no largest triangle.

With $\kappa = -1$, the longest possible length of the sides of a triangle are when the angles $\alpha = \beta = \gamma \to 0$, in which case $\alpha + \beta + \gamma = \pi - A/R^2 = 0$. Therefore, you can get a triangle with infinitely long sides, but the area is πR^2.

5 Brute force:

$$dx = dr\sin\theta\cos\phi + r\cos\theta\,d\theta\,\cos\phi - r\sin\theta\,\sin\phi\,d\phi$$

$$dx^2 = dr^2\sin^2\theta\cos^2\phi + 2 \text{ more squares and } 3 \text{ cross terms}$$

$$dy = dr\sin\theta\sin\phi + r\cos\theta\,d\theta\,\sin\phi + r\sin\theta\,\cos\phi\,d\phi$$

$$dy^2 = dr^2\sin^2\theta\sin^2\phi + 2 \text{ more squares and } 3 \text{ cross terms}$$

$$dz = dr\cos\theta - r\sin\theta\,d\theta$$

$$dz^2 = dr^2\cos^2\theta + r^2\sin^2\theta\,d\theta^2 - 2r\sin\theta\cos\theta\,dr\,d\theta$$

Adding everything together, there are some cancellations, and we end up with

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\theta^2\right)$$