Homework Solutions 2 (Ryden Chapter 3)

3 On a circle, \( r \) is constant, so \( dr = 0 \). Therefore \( dl = R \sin(r/R) \, d\theta \), and the circumference is

\[
C = R \sin\left(\frac{r}{R}\right) \int_0^{2\pi} d\theta = 2\pi R \sin\left(\frac{r}{R}\right)
\]

The deviation from flatness is

\[
\Delta C = 2\pi r - 2\pi R \sin\left(\frac{r}{R}\right) \approx 2\pi r - 2\pi R \left(\frac{r}{R} - \frac{1}{3!} \frac{r^3}{R^3}\right) = \frac{\pi}{3} \frac{r^3}{R^2}
\]

where the approximate value is for \( r \ll R \). To get \( \Delta C > 1 \text{ m} \), we need \( r > 0.0053R = 3.4 \times 10^4 \text{ m} \).

4 With \( \kappa = +1 \), a sphere, the largest triangle is a great circle, with area half the sphere: \( 2\pi R^2 \).

With \( \kappa = 0 \), a plane, there is no largest triangle.

With \( \kappa = -1 \), the longest possible length of the sides of a triangle are when the angles \( \alpha = \beta = \gamma \to 0 \), in which case \( \alpha + \beta + \gamma = \pi - A/R^2 = 0 \). Therefore, you can get a triangle with infinitely long sides, but the area is \( \pi R^2 \).

5 Brute force:

\[
dx = dr \sin \theta \cos \phi + r \cos \theta \, d\theta \cos \phi - r \sin \theta \sin \phi \, d\phi
\]

\[
dx^2 = dr^2 \sin^2 \theta \cos^2 \phi + 2 \text{ more squares and 3 cross terms}
\]

\[
dy = dr \sin \theta \sin \phi + r \cos \theta \, d\theta \sin \phi + r \sin \theta \cos \phi \, d\phi
\]

\[
dy^2 = dr^2 \sin^2 \theta \sin^2 \phi + 2 \text{ more squares and 3 cross terms}
\]

\[
 dz = dr \cos \theta - r \sin \theta \, d\theta
\]

\[
dz^2 = dr^2 \cos^2 \theta + r^2 \sin^2 \theta \, d\theta^2 - 2r \sin \theta \cos \theta \, dr \, d\theta
\]

Adding everything together, there are some cancellations, and we end up with

\[
dx^2 + dy^2 + dz^2 = dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\theta^2 \right)
\]