Homework Solutions 3 (Ryden Chapter 4)

2 In the static case, we have an energy density due to matter, ε_m, and due to the cosmological constant, ε_Λ, perfectly balanced such that with the pressures, $P_m = 0$ and $P_\Lambda = -\varepsilon_\Lambda = -\frac{1}{2}\varepsilon_m$, we get $\dot{a} = \ddot{a} = 0$. In the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon_m + \frac{1}{2}\varepsilon_m - \frac{3}{2}\varepsilon_m\right) = 0$$

Now, instantaneously convert a tiny fraction $\delta \ll 1$ of the matter to radiation. The new energy densities will be $\varepsilon_m' = (1-\delta)\varepsilon_m$ and $\varepsilon_r' = \delta\,\varepsilon_m$, due to energy conservation. The new contribution appearing will be the radiation pressure: $P_r' = \frac{1}{3}\varepsilon_r' = \frac{1}{3}\delta\,\varepsilon_m$. The acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left((1-\delta)\varepsilon_m + \delta\,\varepsilon_m + \frac{1}{2}\varepsilon_m + \delta\,\varepsilon_m - \frac{3}{2}\varepsilon_m\right) = -\frac{4\pi G}{3c^2}\delta\,\varepsilon_m < 0$$

This means $\ddot{a} < 0$, and since we start with $\dot{a} = 0$, it means the universe will contract, due to the extra positive pressure of the photons.

5 The number density behaves like $n = n_0/a^3$. Therefore

$$\varepsilon = nE = \frac{n_0}{a^3}\left(m^2c^4 + \frac{h^2c^2}{\lambda_0^2 a^2}\right)^{1/2}$$

From the fluid equation,

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(1+w)\varepsilon \qquad \Rightarrow \qquad w = -\frac{\dot{\varepsilon}/\varepsilon}{3\dot{a}/a} - 1$$

Taking the derivative,

$$\dot{\varepsilon} = n_0(-3)a^{-4}\dot{a}\left(m^2c^4 + \frac{h^2c^2}{\lambda_0^2 a^2}\right)^{1/2} + n_0 a^{-3}\left(m^2c^4 + \frac{h^2c^2}{\lambda_0^2 a^2}\right)^{-1/2}\frac{h^2c^2}{\lambda_0^2}(-2)a^{-3}\dot{a}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = -\frac{\dot{a}}{a}\left[3 + \frac{h^2c^2}{\lambda_0^2 a^2}\left(m^2c^4 + \frac{h^2c^2}{\lambda_0^2 a^2}\right)^{-1}\right]$$

and

$$w = -\frac{\dot{\varepsilon}/\varepsilon}{3\dot{a}/a} - 1 = \frac{h^2c^2}{3\lambda_0^2a^2}\frac{1}{\left(m^2c^4 + \frac{h^2c^2}{\lambda_0^2a^2}\right)}$$

The nonrelativistic limit is when $mc^2 \gg pc$, and the total energy is dominated by the rest energy. In that limit,

$$w \approx \frac{h^2c^2}{3\lambda_0^2a^2}\frac{1}{(m^2c^4)} \ll \frac{1}{3}$$

As $a \to \infty$, $w \to 0$.

In the relativistic limit, $mc^2 \ll pc$, and the total energy is dominated by the kinetic energy. There,

$$w \approx \frac{1}{3}\left(1 - \frac{m^2c^4}{h^2c^2/\lambda_0^2a^2}\right) \to \frac{1}{3}$$