1 The laws of physics are the same

Just to continue the tradition of misnaming things in physics, what is known as “special relativity” is all about how the laws of physics remain the same in different reference frames.

We’ve taken the invariance of the laws of physics for granted so far: the basic laws of physics, such as those about forces, energy and momentum, and the equations for fundamental forces such as gravity and electromagnetism all remain the same here as on Mars, now as well as a million years ago. In particular, an observer who is not accelerating with respect to us can perform the same experiments to get the same results described by the same forces and so forth. The lack of acceleration is important—for all, \( \sum \vec{F} = m\vec{a} \), and acceleration must be due to new forces. But if we saw a spaceship zipping by at high but constant velocity \( \vec{v} \) relative to us, and if that spaceship was equipped with a lab, they would find that their experiments would produce the same results as those in our labs, and we would all use the same laws of physics to describe what is going on.

But now, here’s a complication. We would expect Maxwell’s equations, which describe electricity and magnetism, to also remain exactly same for our passing spaceship as for us. Both ground-based physicists and physicists in the spaceship would find that Maxwell’s equations have wave solutions and that these waves have a constant speed of \( c \), the speed of light in a vacuum. But that would mean that the speed of light in a vacuum would be \( c \) for any observer. This is seriously weird. Consider what would happen if you tossed a ball with \( v_x = 10.0 \text{ m/s} \), and you had a friend in a car follow the ball in a car going at \( 0.99v_x = 9.9 \text{ m/s} \). You’d expect to observe the ball going at a speed of \( 10.0 \text{ m/s} \) and that your friend should observe a speed relative to them of \( 10.0 - 9.9 = 0.1 \text{ m/s} \). But with light and a rocket ship traveling in the direction of the light, both observers will measure \( c \)!

In particular, if the
light was moving away from you at speed \(c\), and the spaceship was going in the same direction with speed \(0.99c\), the pilot would not measure the speed of that light beam to be \(0.01c\), but still \(c\).

This diagram shows what the velocities look like in the ground frame of reference (you), and the frame of reference of the driver or pilot. You would expect that the picture with the driver should be correct and universal. But all the experimental evidence indicates that it is the picture with the pilot that is correct.

![Diagram showing velocities in different frames of reference](image)

Something has to give. Since speed = distance / time, for the speed of light \(c\) to be a universal constant regardless of the frame of reference, space intervals (distances) and time intervals (durations) must be different in different frames of reference.

## 2 Time dilation

Time intervals depend on the reference frame, such that

\[
\Delta t = \gamma \Delta t_0
\]

Here, \(\Delta t_0\) is “proper time”; it’s the time interval measured in a reference frame where the events defining the beginning and end of the interval take place in the same location. \(\Delta t\) is the dilated (expanded) duration measured in a frame where the beginning and ending events are not at the same location. \(\gamma\) is the “time dilation factor,” which is always \(\gamma = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2} \geq 1\).

For example, let’s say I decided to be ridiculous and speak to you from the back of a truck that was slowly going by a classroom. Let’s also say that as usual, my watch determined the start and end of the class period. The
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beginning event is my looking at my watch and announcing “let’s start.” The ending event is my looking at my watch and proclaiming “we’re done.” My watch would then be measuring the proper time $\Delta t_0$, since in my frame of reference, my watch will not have moved, and therefore it will always be at the same location. My speed relative to myself is zero, after all. Anyone listening to me can also time how long I speak, using their own watch. But they would measure a larger duration $\Delta t$, since the beginning and ending events are no longer at the same location—I will have moved.

Everyday speeds are an extremely small fraction of the speed of light; $v/c \ll 1$. In that case, the time dilation factor $\gamma \approx 1$, and you won’t see the effects of relativity. For example, say the truck I was speaking from went by at an fastish speed of $v = 3.0 \text{ m/s}$. Since $c = 3.0 \times 10^8 \text{ m/s}$, calculate $\gamma$ by using its definition:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

You should get a number that is very close to 1,\(^1\) which means that everyday speeds are much too small for noticeable relativistic effects.

Let’s explore $\gamma$ a bit more. Calculate $\gamma$ for $v = 0$, $v = \frac{1}{4}c$, $v = \frac{1}{2}c$, and $v = \frac{3}{4}c$. Then also find $\gamma$ for $v = 0.99c$ and $v = c$. Using your results, sketch a graph of $\gamma$ vs $v$:

\[\text{Graph}\]

\[^1\gamma = 1.0000000000000005\]
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You should have $\gamma = 1.03, 1.15, 1.51$, and then $\gamma = 7.09$ and $\gamma = \infty$.

As long as we’re close to everyday relative speeds, $\gamma \approx 1$, and we don’t notice any weirdness in space and time. But once $v \approx c$, our everyday intuitions about space and time fail spectacularly.

Let’s try another example. Say there’s a planet 10 $c$-yr away ($c$-yr = a light year; the distance light travels in a year). Say we want to send a spaceship there at extremely high speed, with $v \approx c$ such that $\gamma = 10$.

First, calculate how many years it will take for the spaceship to reach the distant planet in the Earth reference frame, as determined by an observer who remains in Space Control. There will be no relativity involved here; everything is in a single frame of reference, that of the Earth observer. Will this time interval be $\Delta t$ or $\Delta t_0$? The beginning and ending events are the pilot leaving Earth, and the pilot arriving at the same location as the planet. In the Earth reference frame, do these take place at the same location?

Then, figure out the amount of time the journey takes for the pilot. Is this $\Delta t$ or $\Delta t_0$?
In the Earth frame, we will have $\Delta t$: the beginning and ending event are separated by 10 light years! And calculating it is trivial: $\Delta t = 10 \cdot \text{yr}/c = 10$ years.

In the pilot’s reference frame, Earth will be moving away at high speed, and the distant planet will be rushing toward her at close to the speed of light. The beginning and ending events—the pilot being at the same location as the Earth and then the planet, will happen at the same location. So she will measure $\Delta t_0 = \Delta t/\gamma = 1$ year. Yes, the journey will take only one year for her. She will, in fact, age only one year, while those back on Earth age a decade.

3 Length contraction

Space intervals—lengths—also depend on the frame of reference:

$$\Delta x = \frac{\Delta x_0}{\gamma}$$

The proper length $\Delta x_0$ is measured in the frame of reference where the length in question is not moving. In other frames, the length is contracted to $\Delta x$.

If you again imagine me speaking to the class when moving on top of a truck relative to you, I would measure the proper length of the truck, since the truck would be stationary in my frame of reference. You, on the other hand, would, if you took a picture of the truck against a background with a length scale on it, would see the truck as slightly shorter. (Very slightly, since $\gamma \approx 1$ in this case.) By the way: the height of the truck would not change, no matter how fast it was going relative to you. Only lengths parallel to the direction of motion are contracted.

Now, consider the spaceship flying to the planet 10 light years away again. In the pilot’s frame of reference, she is stationary and the planet rushes onto her at almost the speed of light. But we have already established that it takes her only $\Delta t_0 = 1$ year to reach the planet. So her distance to the planet must not be 10 light years! Find $\Delta x_0$ and $\Delta x_0$ in this case, and identify which frame of reference they belong to. Also sketch a picture showing Earth, the distant planet, their speeds, and their distance in the pilot’s frame of reference.
The Earth-planet distance is stationary in the Earth frame, so the given 10 \(c\cdot\text{yr}\) is in fact \(\Delta x_0\). The contracted length \(\Delta x = \Delta x_0 / \gamma = 1 \: c\cdot\text{yr}\) is exactly what you need for the travel time to be 1 year only. The picture:

Note that the planets do not appear spherical in the pilot’s frame, since the length along the direction of motion is contracted to a tenth, while the perpendicular lengths are not affected. The planets look squashed.