Energy and Momentum

If space and time intervals depend on frames of reference, then a lot of other physical quantities must as well. Energy and momentum are the most important ones for our purpose.

1 Relativistic energy

The total energy of an object, when you account for relativity, is

$$E = \gamma mc^2$$

γ is the familiar time dilation factor, m is the mass, and c is the speed of light in a vacuum.

There are a few things we should notice here. First, in a reference frame where an object is at rest, $v = 0$, and therefore $\gamma = 1/\sqrt{1-(v/c)^2} = 1$. We end up with one of the most famous equations in physics, $E = mc^2$. This tells us that mass and energy are actually the same thing. After all, c^2 is just a constant—think of it as a conversion factor between mass units and energy units. And this means that the notion of conserving energy and converting between different forms of energy—kinetic, gravitational, thermal, etc.—applies to mass as well. This is the basis for nuclear energy.

Since c^2 is an enormous number, there is a lot of energy that corresponds to everyday masses. Let's say you have a piece of chalk with mass 0.10 kg. How much energy does that correspond to? Say we use that energy to lift a number of people with average mass 70.0 kg by 1.0 m. What is this number? (Remember that gravitational potential energy is mgy, with $g = 9.8$ m/s^2.) Your result should be seriously large.

The energy corresponding to the chalk's mass has to be divided by the energy needed to lift one 70 kg person:

$$\text{(number of people)} = \frac{(0.1\text{ kg})(3\times10^8\text{ m/s})^2}{(70\text{ kg})(9.8\text{ m/s}^2)(1\text{ m})} = 1.3\times10^{13}$$

This is more than a thousand times the population of the world!

Now, let us return to the fact that when an object is at rest, its total energy is $E = mc^2$. When an object is at rest, its kinetic energy is zero. So what mass is must be a sort of potential energy.

With everyday energies, it can be hard to see how $E = mc^2$ is relevant, just like with everyday speeds, time dilation and length contraction are too small to be observed. But with subatomic particles, we get objects where the potential energies associated with interactions are comparable to their mass times c^2. For example, very large nuclei, high up in the periodic table, are often unstable. Their protons and neutrons are not as tightly bound together as elements lower in the periodic table. So they can break up into smaller nuclei, where the protons and neutrons are more tightly bound. Tightly bound means lower energy—just like when you lower yourself down a flight stairs, you are more tightly bound to the Earth, and your gravitational potential energy is lower. Energy is conserved, so lowering the total energy when the large nucleus breaks up releases energy: the smaller nuclei resulting from the break-up will typically be zipping around with appreciable kinetic energies. But now, since mass is equivalent to energy, you should think of a lowering of energy as also a lowering of mass. And indeed, if you add up the mass of the results of the break-up, you'll end up with a mass that is *lower* than the mass of the original nucleus. This is how nuclear power plants operate: the process of fission—breaking up large unstable nuclei—releases energy and lowers the total mass.

The reverse is also true: if you have the energy to spare, you can create mass. For example, with enough energy, we can create a particle-antiparticle pair. An antiparticle has the same mass and some other basic properties of its corresponding particle, but it has an opposite charge. When they combine, their mass vanishes to produce energy, typically in the form of electromagnetic radiation. Let's say we're doing the reverse: how much energy do we need to create an electron (e^-) and its antiparticle, a positron (e^+)? The mass of an electron is $m_e = 5.11\times10^5$ eV$/c^2$, where an electron volt (eV) is an energy unit: the charge of an electron multiplied by one volt.

Since the positron has the same mass as an electron, we need energy for twice the mass of an electron. Notice that the c^2's will cancel out:

$$2m_e c^2 = 2(5.11 \times 10^5 \text{ eV}/c^2)c^2 = 2(5.11 \times 10^5 \text{ eV}) = 1.022 \text{ MeV}$$

Electron volts (or keV, MeV, GeV, as needed) are convenient units where subatomic particles are concerned, since everyday units are much too large in such contexts.

That is just the minimum energy you need to produce an electron-positron pair—they will be at rest, with no kinetic energy. Anthing extra beyond 1.022 MeV will go into kinetic energy, split equally between the electron and positron moving in opposite directions with the same speed. (An unequal split would violate momentum conservation.)

Since the total energy of a particle at rest is mc^2, but the total energy when it's not at rest is γmc^2, the difference between the two must be the relativistic version of kinetic energy:

$$K = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$$

At very small speeds, where $\gamma \approx 1$, $K \approx \frac{1}{2}mv^2$, which is our more familiar nonrelativistic kinetic energy. But whenever you have speeds close to c or energies close to mc^2, you need to use the relativistic expressions for physical quantities.

Say we put an energy of 2.00 MeV to create an electron-positron pair. At what fraction of the speed of light will the e^- and e^+ be traveling?

Of the energy of 2.00 MeV, 1 MeV will go into the mass and kinetic energy of the e^-, and the other 1 MeV will go into the total energy of the e^+. In other words,

$$1 \text{ MeV} = \gamma m_e c^2 = \gamma(0.511 \text{ MeV}) \qquad \Rightarrow \qquad \gamma = \frac{1 \text{ MeV}}{0.511 \text{ MeV}} = 1.96$$

Knowing γ, we can find v/c:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \qquad \Rightarrow \qquad \frac{v}{c} = \sqrt{1 - \left(\frac{1}{\gamma}\right)^2}$$

In our case, $v/c = \sqrt{1 - (1/1.96)^2} = 0.86$. Our particles will be traveling at 86% of the speed of light. This should't be surprising, since their kinetic energies of $(1 - 0.511) = 0.489$ MeV is comparable to their mass energy of 0.511 MeV.

2 Relativistic momentum

I'm not going to do much with relativistic momentum and momentum conservation, but I should at least mention momentum. Nonrelativistic momentum was $\vec{p} = m\vec{v}$; the relativistic version is $\vec{p} = \gamma m\vec{v}$.

Here's a final question. Say we have a mass m which starts at rest. What is the amount of energy that you need to give the mass to accelerate it to reach the speed of light? Also, recall from your first semester that the momentum transfer to an object, the impulse, is $\Delta\vec{p} = \vec{F}_{\text{average}}\Delta t$. What, then, is the average force or time interval you would need to reach the speed of light? Start with γ for $v = c$, which you calculated when working on the previous pseudolecture.

You would have found that $\gamma = \infty$ for $v = c$. Therefore, the amount of kinetic energy you have to provide is $K = (\gamma - 1)mc^2 = \infty$. You need to supply an infinite amount of energy; therefore, the speed of light is not reachable for an object with $m > 0$.

Similarly, you would need a $\Delta\vec{p} = \infty$, which requires an infinite force or an infinitely long time. Again, you can't accelerate a massive object to reach $v = c$.

But then, what about light itself? Light is made of particles called *photons*. And they clearly travel at c, and their energy is finite. How is this possible? Well,

$$\text{(finite energy)} = \gamma mc^2 = \infty \cdot mc^2 \qquad \Rightarrow \qquad m = 0$$

The photon is a massless particle! Not only can it travel at $v = c$, it cannot travel at any other speed.