Homework Solutions 5 (Ryden Chapter 6 & 7)

6.5 Combining equations 6.27 for $f$ and 6.34 for angular size $\delta \theta$,

$$\Sigma \propto \frac{f}{(\delta \theta)^2} \propto \frac{L}{S_\kappa(r)^2 (1+z)^2 l^2 (1+z)^2} \propto \frac{L}{l^2 (1+z)^4}$$

Since the $S_\kappa(r)$-dependence cancels out, this result is independent of curvature, and is indeed correct for any universe with a Robertson-Walker metric. However, it’s also independent of the acceleration of the expansion of the universe, so it has no information about $q_0$.

6.6 From the relationship between $t_0$ and $t_e$,

$$\delta t_e = \frac{\delta t_0}{1+z} = \frac{1}{2} \text{ day} = 4.32 \times 10^4 \text{ s}$$

Therefore

$$R_{\text{max}} = c \delta t_e = 1.3 \times 10^{13} \text{ m} = 4.2 \times 10^{-4} \text{ pc}$$

Using figure 6.4 with $z = 5$, $d_A \approx 0.3 c/H_0 = 1.3 \times 10^9 \text{ pc}$. And the angular size

$$\delta \theta = \frac{R_{\text{max}}}{d_A} = 3.2 \times 10^{-13} \text{ rad}$$

That should be too small for astronomers to observe.

6.8 Combining equations 5.50, 6.21, and 6.29, we get $f$ for a standard candle,

$$f = \frac{H_0^2 L (1+3w)^2}{16\pi c^2 (1+z)^2 [1 - (1+z)^{-1/(1+3w)})^2]$$

When $w = -\frac{1}{3}$, the integral in equation 5.49 gives a log: $d_p(t_0) = \frac{c}{H_0} \ln(1+z)$, resulting in

$$f = \frac{H_0^2 L}{4\pi c^2 (1+z) \ln(1+z)}$$

Now, $r$ is the proper distance to the standard candle, once again given by equation 5.50. The number of standard candles $r$ to $r+dr$ away is the density $n_0$ times the volume of the spherical shell of radius $r$:

$$dN = \left(4\pi r^2 \, dr \right) n_0$$

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We are looking for those in the range $z$ to $z + dz$. So we use $dr = \frac{dz}{dz} dz$, taking the derivative of equation 5.50. We get

$$dN = \frac{4\pi cn_0}{H_0} d\rho(t_0)^2 (1 + z)^{-3(1+w)/2} dz$$

The intensity is the flux $f$ for each standard candle divided by the full solid angle of $4\pi$ steradians, so

$$dJ = \frac{f}{4\pi} dN = \frac{n_0 Lc}{4\pi H_0} (1 + z)^{(7+3w)/2} dz$$

We integrate this over all $z$ to get the intensity:

$$J = \int_{z=0}^{z=\infty} dJ = \frac{n_0 Lc}{2\pi H_0 (5 + 3w)}$$

As in your exam problem, the horizon distance can diverge when $w < -\frac{1}{3}$, since

$$d_{\text{hor}} = \int_0^\infty dz \frac{dr}{dz} \propto (1 + z)^{-1+3w/2} \left|_0^\infty \right.$$  

But the $J$ integral is different, diverging when $w < -\frac{5}{3}$. In the range of $-\frac{5}{3} < w < -\frac{1}{3}$, the standard candles at large distances are so far redshifted that even the light seen from an infinity of them gives a finite brightness.

**7.4** Equation 7.2 gives $M(r) = v^2 r / G$ for a spherically symmetric halo. The density as a function of distance is the mass $dM$ at radius $r$ divided by volume of a spherical shell at radius $r$, $4\pi r^2 dr$:

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM}{dr} = \frac{v^2}{4\pi G r^2}$$

Mining the book for the relevant numbers, we get $v = 235$ km/s from page 130, which gives

$$\rho(r) = (1.03 \times 10^{-21} \text{ kg/m}^3) \left(\frac{r}{8.2 \text{ kpc}}\right)^2$$

using peculiar units like equation 7.12, where 8.2 kpc is the radius of the Sun’s orbit around the galactic center.

The cosmological constant gives a density $\rho_\Lambda = 0.7 \rho_{c,0}$, which at $r = 8.2$ kpc is about $6 \times 10^{-6}$ of the dark matter density $\rho(r)$. Much farther, at
$r \approx 100$ kpc, the dark energy density is still only about $10^{-3}$ of $\rho(r)$. So while $\Lambda$ is important at cosmological scales, it shouldn’t affect local dynamics.

7.6 Consider a spherical shell of thickness $dr$ at a radius of $r$ from the sun. Neutrinos travel at very nearly $c$ and interact so rarely that we can neglect any absorption. The time it takes a neutrino to cross $dr$ is $dr/c$. The number of solar neutrinos inside that shell, then, is $dN = dr (2 \times 10^{38}/s)/c$. The number density is

$$n = \frac{dN}{dV} = \frac{(2 \times 10^{38}/s) dr}{c 4\pi r^2 dr}$$

At a distance $r = 1$ AU, this gives $n = 2.4 \times 10^6$/m$^3$.

Say a typical human is around 60 kg, and her density is that of water. Their volume is then 0.06 m$^3$. The number of solar neutrinos in that volume is

$$N_s = nV = 1.5 \times 10^5$$

The cosmic neutrino density is in equation 7.49; this gives a number of $N_c = 2.0 \times 10^7$, a lot larger.