Gravity and Black Holes

1 The universe is strange

As we have started to see with special relativity, one of the main lessons from the physics of the twentieth and twenty-first centuries is that far from everyday circumstances, the universe behaves in strange and counterintuitive ways. We have a decent gut-level feel for how objects of everyday sizes, speeds, and temperatures behave under ordinary forces. The math in a physics course can be challenging, but no one struggles to imagine a projectile flying through the air or water cooling to form ice. But when we investigate objects that travel close to the speed of light relative to us, our intuitions about space and time become nearly worthless.

Much of recent physics has been devoted to experimenting with circumstances far from everyday experience, and to devising theories that are still universally reliable. We examine the very large and the very small, or the extremely cold and the extremely hot.

So I want to give you a taste of what physicists have learned. It's unlikely that you'll have to use any of this directly in a professional capacity—the practical significance of this for you is that it may help you talk to a physicist or engineer when you work on a project together. But even more than that, as science majors, I don't want you to think of yourself as people with an extremely narrow expertise. You should know more than your little corner in biology, medical science, or chemistry—you should know something about our universe more generally, and have a context into which you can fit your expertise. As a physicist, sometimes I have to represent not just physics but science in general, and I may need to explain why vaccines are a good idea and that evolution is real. You may, one day, find yourselves explaining that time travel is physically dubious.

First, let's examine the physics of the very large—gravity, black holes, and the universe as a whole.

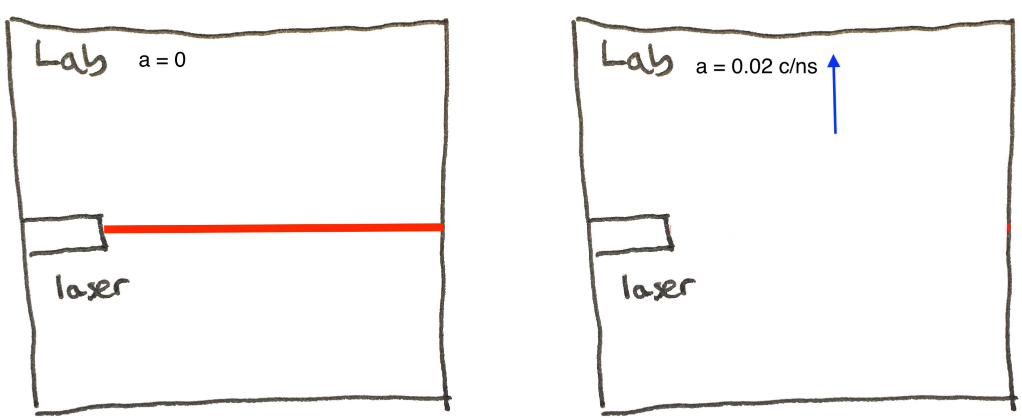
2 General relativity

We calculated that for a photon (a light particle), the time dilation factor $\gamma = \infty$. Now, consider a photon traveling between Earth and a distant planet $10\,c{\cdot}\mathrm{yr}$ away. This $10\,c{\cdot}\mathrm{yr}$ is the proper distance, since it is stationary in the Earth reference frame. In the photon's frame, this distance will be contracted to $(10\,c \cdot \mathrm{yr})/\infty = 0$. And the 10 years it takes for the photon to travel is the dilated time; the proper time in the photon reference frame is $10/\infty = 0$. The photon makes it to its destination in zero time in its frame, because the distance it travels in its frame is zero. It's already at its destination.

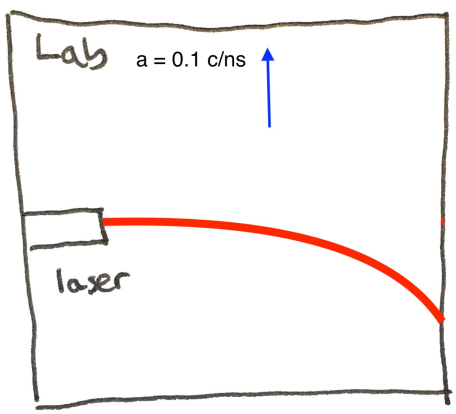
That's strange. But think of its this way: a photon takes the shortest possible distance when traveling. After all, the distance it sees between its origin and destination is zero, and you can't get smaller than that.

What, then, happens, if we try to extend relativity to accelerating frames of reference? In doing so, let's say that photons still behave the same way: they still always travel at a speed c in any frame of reference, and they still take the shortest possible path.

Now, say you're in a well-equipped lab in a spaceship that is not accelerating. You shine a laser beam on the opposite wall, $3\,\mathrm{m} = 10\,c{\cdot}\mathrm{ns}$ away, and measure that the beam has not deflected up or down, and has gone in a perfectly straight line. But then, you turn the engines on, and start accelerating upward at a rate of $a = 0.02\,c/\mathrm{ns}$. Draw what you think the laser beam will look like. If you remember your first semester physics about projectile motion, you can also calculate the spot on the opposite wall the laser beam will hit.



The result will be a parabola, exactly as in projectile motion, where you had motion with a constant vertical acceleration downward and a zero horizontal acceleration. Note that we don't have to deal with any time dilation or length contraction here, since we are not changing our frame of reference at all. As the photon moves toward the opposite wall, the floor of the lab rises at an increasingly fast rate of $0.02\,c/\text{ns}$. It takes 10 ns for it to reach the opposite wall. During that time, the floor will have risen an amount of $\frac{1}{2}a(\Delta t)^2 = \frac{1}{2}(0.02\,c/\text{ns})(10\,\text{ns})^2 = 1\,c{\cdot}\text{ns}$. (Or 0.3 m, if you want.)



First, notice that in the lab frame, the path of the photon is curved. And still, the photon takes the shortest path. This means that the underlying spacetime is not flat—if it were flat, the shortest path would be a straight line. Spacetime itself must be curved, the way we live on the surface of a spherical planet, and when we fly from Los Angeles to New Delhi, we do not travel in a straight line but over the arc of a "great circle" on the Earth. A curved two-dimensional surface, like a sphere, is just about within our capability to visually picture. But a curved 3+1-dimensional (3 space and 1 time) spacetime geometry is beyond our capability to visualize.

Second, in the lab frame, dropped objects accelerate *downward toward the floor* at a rate of $a = 0.02\,c/\text{ns}$. The picture with $\vec{a}$ pointing upward is the view from an outside frame, say ground control on Earth. In the lab, the situation is indistinguishable from what happens with a downward gravitational force. After all, when we drop objects, the accelerate downward at 9.8 m/s^2. We feel the normal force of the floor on our feet, preventing us from accelerating toward the center of the Earth. Gravity and acceleration are indistinguishable; therefore light beams will curve in the presence of gravity as well. The statement that an object is affected by gravity is the same as saying that the path of objects follow the local curvature of spacetime. And since gravity is due to mass, and in relativity mass and energy are the same ($E = mc^2$), the curving of space and time must be due to the local energy density. The more energy you pack into a volume, the bigger the curvature and the stronger the gravity.

General relativity, then, is a theory of gravity. A common slogan physicists use to describe general relativity is that **mass (energy) tells spacetime how to curve, and curvature tells mass (energy) how to accelerate.**

Unless you go to extremes, our familiar Newtonian gravity, with an attractive force $F_G = G\, m_1 m_2/r^2$, works fine. You need seriously large gravity (equivalently, acceleration) to notice any difference. In our example we had an absurdly high $a = 0.02\, c/\text{ns} = 6 \times 10^{15}\, \text{m/s}^2$; compare this to our familiar $g = 9.8\, \text{m/s}^2$. You won't see laser beams curving in everyday life. If we had an acceleration $a = 6 \times 10^{15}\, \text{m/s}^2$ in a spaceship, its passengers would be totally crushed and flattened.

3 The radius of a black hole

Let's investigate some extremes. A popular favorite is a black hole, where the gravity is so strong not even light can escape.

The math required for calculations in general relativity is very difficult. But we can still make an approximate, Newtonian calculation for a black hole. And as luck would have it, the errors we make cancel each other out, and we can easily derive an exact result for the radius of the event horizon of a black hole.

Since the key to a black hole is that not even light can escape its gravity, let's start with a question. Say we're standing on a spherical planet in space, which has no atmosphere to complicate matters. The mass of the planet is M, and its radius is R. How fast do we have to toss a rock, with mass m, straight up so that it will never fall back? This is the *escape speed* of that planet, and it is easily calculated by using energy conservation. After all, just after we toss the rock, it will have a total energy: the kinetic energy we just gave it, and the gravitational potential energy that depends on its distance r to the center of the planet. And then, there's the total energy when the rock has escaped: zero kinetic energy (it only had enough to barely escape) and the gravitational potential energy when it's infinitely far away. These total energies should be the same.

The kinetic energy of the rock is $\frac{1}{2}mv^2$. This is one of the nonrelativistic approximations we make; one of the errors that fortuitously cancels out. In any case, the initial speed v_i is what we are trying to find, and the final speed $v_f = 0$.

The gravitational potential energy is

$$U_G = -G\,\frac{mM}{r}$$

Note, first of all, that the gravitational potential energy is negative. The energy is smaller (more negative) as r becomes smaller. The approximation we use when close to the surface of the Earth behaves the same way: mgy drops as y becomes smaller. Second, $U_G \propto \frac{1}{r}$, just like the electrical potential energy for a point charge $U_E \propto \frac{1}{r}$. Both Newtonian gravity and electric forces are inverse-square forces, so this similarity is not an accident—the math is very similar.

So, do this calculation. Write down the total energy at the initial state, with an unknown v_i and a known $r = R$. Then equate that to the final total energy, where $v_f = 0$ and $r \to \infty$. Then solve for v_i. That's your escape speed from any spherical mass distribution with mass M and radius R. If you toss your rock at that speed or more, it will escape. Less speed, and it will eventually fall back down. (Assuming nothing else in space affects it, and so forth.)

Here's how it works:

$$\frac{1}{2}mv_i^2 - G\frac{mM}{R} = \frac{1}{2}m0^2 - G\frac{mM}{\infty} = 0 \quad \Rightarrow \quad \frac{1}{2}mv_i^2 = G\frac{mM}{R}$$

Solving for v_i, and giving it its proper name, we end up with

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

And now for the black hole. Set the escape speed of our spherical mass equal to the speed of light. Since no signal can travel faster than light, this sets the limit. It's best to solve for the radius R:

$$c = \sqrt{\frac{2GM}{R}} \qquad \Rightarrow \qquad R = \frac{2GM}{c^2}$$

The radius R in this case is known as the "event horizon." At the event horizon—which is not a solid surface or anything; it's just an abstract boundary in space—only light speed signals can get out. Closer to the black hole than the event horizon, *nothing* can get out. The event horizon is a point of no return.

Just to get an idea, let's calculate the event horizon radius for the Earth, and compare it to the actual radius of the Earth. We would have to compress Earth's mass into a sphere of that radius to turn it into a black hole. You can look up G and the mss of Earth and the speed of light and calculate this. While you're at it, go ahead and calculate the mass density of the Earth-mass black hole, and compare it to the density of water, which you can also look up.

Here's the calculation:

$$R = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11})(5.97 \times 10^{24})}{(3.00 \times 10^8)^2} = 8.85 \times 10^{-3}\,\mathrm{m}$$

Just under a centimeter! The actual radius of the earth is 6.38×10^6 m, much larger.

The density, compared to the density of water at 1000 kg/m^3, is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = 2.06 \times 10^{30}\,\mathrm{kg/m^3} \gg 10^3\,\mathrm{kg/m^3}$$

This is incredibly large. Small back holes (Earth is small by cosmic standards) have enormous densities.

How can we create black holes, since we need to compress mass down into such huge densities? One way is as a result of the death of large (about 20 solar masses or so) stars. Stars keep shining by fusing Hydrogen into Helium—a star is a massive ongoing thermonuclear explosion. Eventually their fuel runs out, and the pressure of the incredibly hot gas that is a star drops. That pressure keeps the star going, acting to cancel out the enormous inward gravitational force that would otherwise cause gravitational collapse. When out of fuel, the star collapses, and if it's massive enough, the result can be a black hole. Astrophysicists have detected such black holes, along with much larger supermassive black holes at the centers of most galaxies.