1. **(30 points)** You have the following circuit, where the 8.0 Ω resistance represents an appliance. This appliance has two modes of operation: low power and high power. The 3.0 V battery on the same branch of the circuit as the appliance can be connected as shown, or it can be connected the opposite way, with the + and − terminals reversed. Calculate the power dissipated by the appliance in both modes.

![Circuit Diagram]

**Answer:** First, let’s analyze the circuit as shown.

Draw current arrows: call the one through the 1 Ω resistor $I_1$ (to the right), the 2 Ω resistor $I_2$ (downward), and the 8 Ω resistor $I_3$ (to the right). Put + signs on the side of the resistors where the arrow bases are, − signs at the ends the arrows point toward.

There is just one independent junction equation, $I_1 = I_2 + I_3$. Setting the rises equal to the drops on the left loop produces $9 \text{ V} = V_1 + V_2$. For the right loop, $V_2 = V_3 + 3 \text{ V}$. Putting in the resistor $V = RI$ relations turns the loop equations into $9 \text{ V} = (1 \Omega)I_1 + (2 \Omega)I_2$ and $(2 \Omega)I_2 = (8 \Omega)I_3 + 3 \text{ V}$.

These are three equations for three unknown currents. We only care about the current through the appliance, which is $I_3$. Solving the equations, we get $I_3 = \frac{9}{26} \text{ A} = 0.35 \text{ A}$.

In that case, the power dissipated by our appliance is $P_3 = V_3I_3 = (8 \Omega)I_3^2 = 0.96 \text{ W}$. This is the low power mode.

The high power mode just inverts the 3 V battery. So our equations are the same, except that wherever we had the original 3 V, we need to replace it with $-3 \text{ V}$: $I_1 = I_2 + I_3$ and $9 \text{ V} = (1 \Omega)I_1 + (2 \Omega)I_2$ remain the same, but the right loop equation becomes $(2 \Omega)I_2 = (8 \Omega)I_3 - 3 \text{ V}$.

Solving these, we get a new $I_3 = \frac{27}{26} \text{ A} = 1.04 \text{ A}$.

The power dissipated by our appliance is $P_3 = V_3I_3 = (8 \Omega)I_3^2 = 8.63 \text{ W}$. 
2. (50 points) In the classroom, we discussed the following circuit for discharging a capacitor. The capacitor is fully charged at first, with voltage \( V_0 \) across it, when the switch is open; when the switch is closed at time \( t = 0 \), the charge starts to decline. We also discussed that since the current in the circuit after the switch is closed is due to the charges on the capacitor plates moving away, that the larger the current, the faster the capacitor will be discharged. Since a larger resistance \( R \) means a smaller current, the capacitor will take longer to discharge, with the current through the capacitor \( I_C(t) = I_C(0) e^{-t/RC} \) and \( I_C(0) = V_0/R \).

\[
\begin{align*}
&\text{\scriptsize Circuit Diagram} \\
&C \quad \text{\textasciitilde} \quad R
\end{align*}
\]

Now say you have two identical resistances, both \( R \), which you hook up (1) in series, and (2) in parallel, and you replace your original \( R \) with these two two-resistor configurations.

(a) Draw a diagram for each circuit, (1) and (2). Write down junction and loop equations for each circuit.

**Answer:**

\[
\begin{align*}
&(\text{Series Case}) \\
&C \quad \text{\textasciitilde} \quad R \quad \text{\textasciitilde} \quad R

&(\text{Parallel Case}) \\
&C \quad \text{\textasciitilde} \quad R \quad \text{\textasciitilde} \quad R
\end{align*}
\]

In the series case, there are no junctions and a single current \( I_C \), and the loop equations gives \( V_C = 2RI_C \).
In the parallel case, we have $I_C = I_1 + I_2$ and the loop equations $V_C = V_1$ and $V_1 = V_2$. Putting these together, $RI_1 = RI_2 = V_C$, and therefore $I_1 = I_2$ and $I_C = 2I_1$.

(b) Find expressions for $I_{C1}(0)$ and $I_{C2}(0)$, the currents through the capacitor at time $t = 0$ for each circuit, in terms of variables such as $V_0$, $R$, and $C$. Which circuit, therefore, will discharge faster?

**Answer:** Using the loop and junction equations, for the series case, at $t = 0$ we start with $V_C = V_0 = 2RI_C$, therefore $I_C(0) = V_0/2R$. For the parallel case, $I_C(0) = 2V_0/R$.

Since the current is higher for the parallel case, it will discharge faster.

(c) Compare your result in (b) to the single-$R$ circuit. All that has changed is the effective resistance values that appear in $I_C(0)$; after all, $C$ is the same. So, by analogy, write down expressions for $I_{C1}(t)$ and $I_{C2}(t)$, for all times $t$. (Ask me for help if you need it!) Qualitatively sketch a graph of these two currents versus time, representing both currents on the same graph.

**Answer:** When comparing to the single resistor case, where $I_C(0) = V_0/R$, we can see that the effective resistance for the series case is $2R$, while for the parallel case it is $R/2$. The times scales for the exponential discharge, then, will be $2RC$ and $RC/2$, with expressions

$$I_{C1}(t) = \frac{V_0}{2R} e^{-t/2RC} \quad \text{and} \quad I_{C2}(t) = \frac{2V_0}{R} e^{-2t/RC}$$

When you graph these, the exponential decay of $I_{C1}(t)$ will start from an initial value that is four times that of $I_{C2}(t)$, but it will also fall four times as fast. Those of you who remember a bit of calculus might note that the area under both of these curves is the total charge of the capacitor, and hence the same.

3. **(30 points)** You have the following circuit.
Draw the current in this circuit, and then draw the magnetic field produced by the wire segment on the opposite side of the square around each side of the square, using \( \cdot \) and \( \times \) symbols as appropriate. Note: This is not the same as drawing in the total magnetic field produced by the circuit!

Now, also draw in the forces exerted on each wire segment by its opposite-side wire in the square. (These are the most important forces.)

What do you think the total magnetic force exerted by the circuit on itself is? Draw in the total force as well. Label everything carefully.

**Answer:**

The current through the circuit will be clockwise. Using the right hand rule, you can see that the current in each wire will produce a magnetic field into the page in the vicinity of its opposite wire. Therefore the magnetic force felt by each wire will be outward. Another way of seeing this is that opposite wires have antiparallel currents, which repel one another.

The total force is zero. All these forces are action-reactions pairs; a body
can’t produce a total force that it exerts on itself. (You can’t lift yourself by your bootstraps.)

4. (50 points) You have three circuits, a, b, c. Circuits a and b have the same resistor $R_1$ and a power source. Circuit a has a resistor $R_1$ and a DC power source with a constant $V = V_0$. Circuit b is identical except that it has an AC power source, $V = V_0 \cos(2\pi ft)$ with frequency $f = 60$ Hz. Circuit c just has a resistor $R_2$.

You now have two situations, one where circuit a is close to c, and in the other, b:

(a) For both of these cases, discuss whether a current will be induced in circuit c. Explain your answers, giving all relevant physical details.

Answer: When a and c are brought together, there will be no current induced in c. With a DC power source in a, the current in the circuit is constant. The magnetic field created by the current in a will be constant, and hence the magnetic flux through c will be constant. Since the flux is not changing, no voltage will be induced in circuit c, and therefore no current.

In contrast, circuit b has an AC power source, which produces a voltage that changes in time. Therefore the current in b also varies sinusoidally with frequency $f$. And since the magnetic field produced by a current is proportional to the current, the magnetic field magnitude will also vary with frequency $f$. The result is a changing magnetic flux through c, which produces an alternating current in c.

(b) Given your answer to (a), think about the following problem. You have multiple electrical appliances in a room—for example, medical
devices. You are concerned about them interfering with each others’ operations. Evaluate each of the following options, including a short explanation of why it is a helpful or a useless idea:

- Make sure all devices operate on DC power sources, such as batteries.

  **Answer:** This would obviously help: any change would be confined to the time in which the devices are being turned on and off.

- Make sure all devices operate on AC power sources, such as household electricity.

  **Answer:** This is not a good idea, as seen in the answer to (a). AC means changing magnetic fields, hence changing magnetic fluxes, and interference.

- Appliances are often encased in metal. Recall our lab experiment where the electric field did not penetrate inside a metal ring. So encase all devices in metal to make sure their circuits are shielded from external electric fields, preventing interference.

  **Answer:** This is irrelevant. Metal does not shield from magnetic fields, and so any interference due to changing magnetic fluxes would remain.

- Keep the distances between your devices as large as possible.

  **Answer:** Since the magnetic fields produced by currents become smaller in magnitude as you go further away, larger distances also reduce the size of the rate of change of magnetic flux. Therefore this would help.

- Make sure the areas enclosed by each circuit are as small as possible.

  **Answer:** This helps a little, since it reduces the flux, and therefore also the rate of change of magnetic flux. Here’s one way to
think about it. Since the area is not changing, \( \frac{d}{dt} \Phi = A \frac{d}{dt} B_\perp. \) All else being equal, reducing the multiplier \( A \) will reduce the rate of change of \( \Phi \).

- Make sure to design circuits such that their operation is not sensitive to low level electrical noise.

**Answer:** Well, obviously.

5. **(40 points)** You accelerate an electron, starting from rest in the lab, by applying a voltage of \( 5.11 \times 10^5 \) V, which is not too difficult in a decent lab. (Note that the mass of an electron is \( 5.11 \times 10^5 \) eV/\( c^2 \).)

(a) Calculate the speed of this electron using the nonrelativistic form of kinetic energy, which you have been using in all circumstances until you encountered relativity.

**Answer:** Energy conservation dictates \( eV = \frac{1}{2} m_e v^2 \). Therefore, and remembering that \( e \) times a Volt is an eV,

\[
v = \sqrt{\frac{2eV}{m_e}} = \sqrt{\frac{2e(5.11 \times 10^5 \text{ V})}{5.11 \times 10^5 \text{ eV}/\text{c}^2}} = \sqrt{2} c = 4.2 \times 10^8 \text{ m/s}
\]

Note that this is faster than light!

(b) Calculate the speed of this electron again, now using the relativistic form of kinetic energy, which you have only encountered very recently.

**Answer:** Since now \( eV = (\gamma - 1)m_e c^2 \), we can see that

\[
e(5.11 \times 10^5 \text{ V}) = (\gamma - 1)(5.11 \times 10^5 \text{ eV}/\text{c}^2) c^2 \quad \Rightarrow \quad \gamma = 2
\]

We can now solve for \( v \), since \( \gamma = 1/\sqrt{1 - (v/c)^2} \).

\[
v = \sqrt{1 - \frac{1}{\gamma^2}} c = \sqrt{\frac{3}{4}} c = 0.87 c
\]

Properly below the speed of light!
(c) Which result do you trust more? Explain.

**Answer:** You should *always* trust the relativistic calculation. When $v \ll c$, nonrelativistic expressions such as $\frac{1}{2}mv^2$ are approximately correct, but otherwise—at high speeds and high energies—they become useless.

(d) In the lab, this electron travels a distance of 30.0 m in the lab before hitting a target that is stationary. Find out what this distance is in the reference frame in which the electron is at rest.

**Answer:** Since the endpoints of the length are stationary in the lab, $\Delta x_0 = 30$ m is the proper length. In the electron’s frame, the distance is $\Delta x = \Delta x_0/\gamma = 30/2 = 15$ m.

(e) What is the time it took for the electron to travel to the target in the lab frame? In the electron’s frame?

**Answer:** In the lab frame,

$$\Delta t = \frac{\Delta x_0}{v} = \frac{(30 \text{ m})}{(0.87 c)} = 1.2 \times 10^{-7} \text{ s}$$

In the electron’s frame,

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \frac{\Delta x}{v} = 5.8 \times 10^{-8} \text{ s}$$