1. **(50 points)** A black hole’s event horizon defines a surface in space; no information about what is within that surface can be transmitted to the outside world. Since entropy is a measure of missing information, it won’t surprise you that the entropy of a black hole is proportional to its area. In fact, with the Planck length \(L_p = \sqrt{\frac{G\hbar}{c^3}}\),

\[
\frac{S}{k} = \frac{A}{4L_p^2} = \frac{c^3}{4G\hbar} A
\]

where \(A\) is the surface area. In the following, recall that a non-rotating Schwarzschild black hole has radius \(R = \frac{2GM}{c^2}\), where \(M\) is the mass of the black hole. Also, in your results, the only physical constants that appear should be \(k\), \(G\), \(\hbar\), and \(c\), and please simplify your results as much as possible.

*Note: This is a classic question, which means you can find almost all the solutions to the following online. Nonetheless, I expect you to do this yourself, without consulting online sources. Consult me if you have any questions; I will make sure you’re on the right track.*

(a) The energy of a black hole is \(U = Mc^2\). Calculate the temperature of a Schwarzschild black hole with mass \(M\).

**Answer:** Putting things together,

\[
S = \frac{4\pi kG}{\hbar c} M^2 = \frac{4\pi kG}{\hbar c^5} U^2
\]

\[
\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{8\pi kG}{\hbar c^5} U = \frac{8\pi kG}{\hbar c^3} M \quad \Rightarrow \quad T = \frac{\hbar c^3}{8\pi kG} \frac{1}{M}
\]

Note that the smaller the black hole, the higher the temperature.

(b) A black hole is a perfect blackbody, which will radiate energy due to its temperature. Calculate the rate of evaporation of a Schwarzschild black hole of mass \(M\). Assume that only photons will be radiated. Note that the Stefan-Boltzmann constant \(\sigma = \pi^2 k^4 / 60\hbar^3 c^2\).

**Answer:** The power radiated by a blackbody is \(\sigma eAT^4\). For a perfect blackbody \(e = 1\). Since there’s no compression-expansion work in outer space, \(dU = dQ\). Therefore

\[
\frac{dU}{dt} = c^2 \frac{dM}{dt} = -\sigma AT^4
\]
The negative sign is because the energy/mass of the black hole will decrease with the evaporation. Putting in everything,

\[ \frac{dM}{dt} = -\frac{\hbar c^4}{2^{10}15\pi G^2 M^2} \frac{1}{M^2} = -r \frac{1}{M^2} \]

where I’ve called the constant up front \( r \).

(c) Calculate \( t_{ev}(M) \), the time it takes for a Schwarzschild black hole to totally evaporate. Calculate this value for a typical stellar remnant black hole with \( M = 20M_\odot \). (\( M_\odot \) is a solar mass.)

**Answer:** Rewrite the rate as \( M^2 dM = -r dt \). We can integrate both sides, starting from \( t = 0 \) where the mass is \( M \) and going until \( t_{ev} \) where the mass is 0:

\[ \int_{M}^{0} dM \frac{M^2}{t_{ev}} = -r \int_{0}^{t_{ev}} \Rightarrow \quad t_{ev} = \frac{M^3}{3r} = \frac{2^{10}5\pi G^2}{\hbar c^4} M^3 \]

Putting in the numbers for \( M = 20M_\odot \), we get \( t_{ev} = 5.3 \times 10^{78} \) s, or \( 1.7 \times 10^{71} \) years. A lot longer than the age of the universe!

2. **(50 points)** You have a heat engine that uses a monatomic ideal gas as a working fluid, and that goes through the following cycle, where \( P_2 = 2P_1 \) and \( V_3 = 2V_1 \).
(a) Calculate the efficiency of the heat engine. Note, however, that you should not assume that the heat transfer to the engine during 2 → 3 is always the same sign. Find out if there are regions on the line segment where $Q < 0$, and if so, account for it as part of $Q_h$ or $Q_c$ accordingly.

**Answer:** For $1 \rightarrow 2$,

$$W_{12} = 0 \quad Q_{12} = \Delta U - W = C_v \Delta T = \frac{3}{2} (P_2 V_1 - P_1 V_1) = \frac{3}{2} P_1 V_1 > 0$$

For $3 \rightarrow 1$,

$$W_{31} = -P \Delta V = P_1 V_1 \quad Q_{31} = \Delta U - W = C_p \Delta T = \frac{5}{2} (P_1 V_1 - P_1 V_3) = -\frac{5}{2} P_1 V_1 < 0$$

For $2 \rightarrow 3$, the equation of the path is

$$P(V) = P_1 \left(3 - \frac{V}{V_1}\right)$$

Therefore, since $PV = NkT$, on this path,

$$P_1 \left(3 - \frac{V}{V_1}\right) V = NkT \quad \Rightarrow \quad P_1 \left(3 - \frac{2V}{V_1}\right) dV = Nk dT$$

Now, since $dW = -pdV$ and $dQ = dU - dW$,

$$dQ = \frac{3}{2} Nk dT + PdV = P_1 \left[\frac{3}{2} \left(3 - \frac{2V}{V_1}\right) + \left(3 - \frac{V}{V_1}\right)\right] dV$$

$$= \frac{P_1}{2} \left[15 - 8 \frac{V}{V_1}\right] dV$$

For $V$ increasing from $V_1$ until $2V_1$, $dV > 0$, and this means that $dQ > 0$ for $V_1 < V < \frac{15}{8} V_1$, and $dQ < 0$ for $\frac{15}{8} V_1 < V < 2V_1$. The two line segments must be accounted for separately, as part of $Q_h$ and as part of $Q_c$. The positive and negative parts are:

$$Q_+ = \frac{P_1}{2} \int_{V_1}^{\frac{15}{8} V_1} dV \left(15 - 8 \frac{V}{V_1}\right) = \frac{49}{32} P_1 V_1 = 1.53 P_1 V_1$$

$$Q_- = \frac{P_1}{2} \int_{\frac{15}{8} V_1}^{2V_1} dV \left(15 - 8 \frac{V}{V_1}\right) = -\frac{1}{32} P_1 V_1 = -0.03 P_1 V_1$$
The work done from $2 \rightarrow 3$ is straightforward; $W_{23} = -\frac{3}{2} P_1 V_1$.

Adding positive and negative heats up separately,

\[
Q_h = Q_{12} + Q_+ = \frac{97}{32} P_1 V_1 = 3.03 P_1 V_1
\]

\[
Q_c = -Q_{12} - Q_- = \frac{81}{32} P_1 V_1 = 2.53 P_1 V_1
\]

\[
W = -W_{12} - W_{23} - W_{31} = \frac{1}{2} P_1 V_1 = 0.50 P_1 V_1
\]

The efficiency is

\[
e = \frac{W}{Q_h} = \frac{16}{97} = 0.165
\]

(b) Find the efficiency of a Carnot engine operating between the maximum and minimum temperatures on this cycle.

**Answer:** We can start by remembering that isotherms are hyperbolas, and that the hyperbolas move away from the axes as $T$ increases. Therefore, from symmetry, the maximum temperature on the cycle is midpoint on the $2 \rightarrow 3$ part. This is at $P = \frac{3}{2} P_1$ and $V = \frac{3}{2} V_1$, so that

\[
T_h = \frac{9 P_1 V_1}{4 Nk}
\]

The minimum temperature is at point 1, so $T_c = P_1 V_1/Nk$. The maximum efficiency is therefore

\[
e_{\text{max}} = 1 - \frac{T_c}{T_h} = \frac{5}{9} = 0.556
\]