Homework Solutions 7a (Schroeder Chapter 7)

17

(a) The diagrams for bosons just indicate the occupancy of each level. The ground state is occupied by \( N \) particles minus the total number of particles pushed up into excited states. So without this ground state, the diagrams are like:

\[
\begin{array}{cccccc}
q = 0 & q = 1 & q = 2 & q = 3 & q = 4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 2 & 1 \\
\end{array}
\]

(b) There are 11 possible \( q = 6 \) states. The average occupancy for the excited states is then the result of adding up the numbers across each row and dividing by 11. The ground state has occupancy \( N - \frac{35}{11} \). With the ground state energy set to zero, here is a plot:

![Plot](image.png)
(c) At $\epsilon = \mu$, $\bar{n} \to \infty$. If $N$ is large, this means $\mu \approx 0$. The plot above fits the Bose-Einstein distribution for the occupancy graph at $kT = 2.2\eta$. Or, you can say that at $\epsilon = \mu + kT$, the occupancy should be $1/(\epsilon - 1) \approx 0.6$, and from the data points it looks like this is just above $\epsilon = 2\eta$.

(d) The entropy values turn out to be the same as the fermionic problem, 7.16, I did as an example:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\Omega$</th>
<th>$S/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.69</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.10</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>1.61</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>1.95</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>2.40</td>
</tr>
</tbody>
</table>

And approximating the slope for $1/kT \approx \Delta S/\Delta U$ again gives $kT \approx 2.2\eta$.

24 The neutron star is a bit simpler than the white dwarf. The only particles are neutrons, and $N = M/m_n$. The kinetic energy is then

$$K = \frac{3}{5}NE_F = \frac{3\hbar^2}{40m_n} \left( \frac{M}{m_n} \right)^{5/3} \left( \frac{9}{4\pi^2R^3} \right)^{2/3} = \frac{\beta}{R^2}$$

with the constants being packed into $\beta$. The gravitational potential energy is the same as the white dwarf, so

$$U = K + V = \frac{\beta}{R^2} - \frac{\alpha}{R}$$

with $\alpha = (3/5)GM^2$. Minimizing the energy with $dU/dR = 0$ gives, for a one solar mass neutron star,

$$R = \frac{2\beta}{\alpha} = (0.093) \frac{\hbar^2}{Gm_n^{8/3}M^{1/3}} = 1.23 \times 10^4 \text{ m}$$

Its density is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = 2.6 \times 10^{17} \text{ kg/m}^3$$
This is comparable to the density of an atomic nucleus. The Fermi temperature is

\[ T_F = \frac{1}{k} \frac{h^2}{8m_n} \left( \frac{3N}{\pi V} \right)^{2/3} = \frac{h^2}{8km_n^{5/3}} \left( \frac{9M}{4\pi^2} \right)^{2/3} \frac{1}{R^2} = 6.6 \times 10^{11} \text{ K} \]

This is higher than a white dwarf. In any case, the assumption that \( T \ll T_F \) is safe.

A neutron star becomes relativistic and unstable when the average neutron kinetic energy is comparable to \( m_n c^2 = 940 \text{ MeV} \). The average \( K = \frac{3}{2} kT_F = 34 \text{ MeV} \) for a one solar mass neutron star, which is too small by \( 1/28 \). But since \( E_F \propto M^{4/3} \), to get into the relativistic regime, we would need \( 28^{3/4} = 12 \) solar masses. This is a crude calculation; astrophysicists calculate the real critical mass at about 2 to 3 solar masses.

### 34

(a) Say \( g_{0v} < g_{0c} \). In that case, if \( \mu \) were to remain constant, there would be fewer electrons added to the conduction band than removed from the valence band. That can’t happen, and therefore \( \mu \) would have to decrease with increasing \( T \). If \( g_{0v} > g_{0c} \), \( \mu \) will have to increase with \( T \).

(b) The number of conduction electrons is

\[ N_c = \int_{E_v}^{\infty} dE \ g(E) \bar{n}_{FD}(E) = g_{0c} \int_{E_c}^{\infty} dE \ \frac{\sqrt{E - E_c}}{e^{(E - \mu)/kT} + 1} \]

with \( kT \ll E_v - \mu \),

\[ N_c \approx g_{0c} \int_{E_c}^{\infty} dE \ \frac{\sqrt{E - E_c}}{e^{(E - \mu)/kT}} = \frac{g_{0c}(kT)^{3/2}}{2 e^{(E_v - \mu)/kT}} \int_0^{\infty} dx \ x^{1/2} e^{-x} = \frac{\sqrt{\pi} g_{0c}(kT)^{3/2}}{2 e^{(E_v - \mu)/kT}} \]

(c) The occupancy for holes is \( 1 - \bar{n}_{FD} \). Therefore

\[ N_h = \int_{-\infty}^{E_v} dE \ g(E) \ [1 - \bar{n}_{FD}(E)] = g_{0v} \int_{-\infty}^{E_v} dE \ \frac{\sqrt{E_v - E}}{e^{(\mu - E)/kT} + 1} \]

This is much like before. When \( kT \ll \mu - E_v \), we get

\[ N_h \approx g_{0v} \int_{-\infty}^{E_v} dE \ \frac{\sqrt{E_v - E}}{e^{(\mu - E)/kT}} = \frac{g_{0v}(kT)^{3/2}}{2 e^{(\mu - E_v)/kT}} \int_0^{\infty} dx \ x^{1/2} e^{-x} = \frac{\sqrt{\pi} g_{0v}(kT)^{3/2}}{2 e^{(\mu - E_v)/kT}} \]
(d) $N_h = N_e$ must be true at any $T$. Therefore,

$$\frac{g_{0c}}{g_{0v}} = \frac{e^{(E_c-\mu)/kT}}{e^{(\mu-E_v)/kT}} = e^{(E_v+E_c-2\mu)/kT}$$

Taking the log, we find how $\mu$ deviates from the center point between the two bands:

$$\mu = \frac{1}{2}(E_v + E_c) + \frac{1}{2}kT \ln(g_{0v}/g_{0c})$$

(e) For Si, $\ln(0.44/1.09) = -0.91$. Therefore $\mu$ shifts by about $\frac{1}{2} \frac{1}{40} (-0.91) = -0.012$ eV. This is considerably smaller than the band gap of 1.1 eV, so the approximations made should work.

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(a) Energy density:

$$\frac{U}{V} = \frac{8\pi^5}{15} \frac{(kT)^4}{(hc)^3} = 0.855 \text{ J/m}^3$$

(b) Spectrum:

(c) Visible light goes from about 1.77 eV to 3.1 eV. With $kT = 0.5$ eV, we have to evaluate

$$\int_{1.77/0.5}^{3.1/0.5} dx \frac{x^3}{e^x - 1}$$
and divide this by the full energy in the spectrum,
\[
\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}
\]

Done numerically, this results in 0.37.

44

(a) To get the number of photons, we need to sum the occupancy of each state. With a factor of 2 to account for the two possible photon spin states,
\[
N = 2 \sum_{n_x, n_y, n_z} \frac{1}{e^{\frac{\hbar c n}{2LkT}} - 1}
\]

As in the text, approximate this as an integral in spherical coordinates (so the eighth of a sphere gives \( \frac{\pi}{2} \)), and this gives you the desired result. If you then numerically evaluate the integral, you get
\[
N = (2.404) 8\pi V \left( \frac{kT}{\hbar c} \right)^3
\]

(b) With equation 7.89, we get \( \frac{S}{N} = 3.60 k \)

(c) The photon density is
\[
\frac{N}{V} = (2.404) 8\pi \left( \frac{kT}{\hbar c} \right)^3
\]

For 300 K, this gives \( 5.5 \times 10^{14}/m^3 \). For 1500 K, we get \( 6.8 \times 10^{16}/m^3 \). And for 2.73 K, we end up with \( 4.1 \times 10^8/m^3 \). The cosmic microwave photon density in the universe is much larger than the density of matter overall.