
Thermo & Stat Mech Exam 1 Solutions

1. (50 points) A black hole's event horizon defines a surface in space; no information about what is within that surface can be transmitted to the outside world. Since entropy is a measure of missing information, it won't surprise you that the entropy of a black hole is proportional to its area. In fact, with the Planck length $L_p = \sqrt{G\hbar/c^3}$,

$$\frac{S}{k} = \frac{A}{4L_p^2} = \frac{c^3}{4G\hbar} A$$

where A is the surface area. In the following, recall that a non-rotating Schwarzschild black hole has radius $R = 2GM/c^2$, where M is the mass of the black hole. Also, in your results, the only physical constants that appear should be k , G , \hbar , and c , and please simplify your results as much as possible.

Note: This is a classic question, which means you can find almost all the solutions to the following online. Nonetheless, I expect you to do this yourself, without consulting online sources. *Consult me if you have any questions;* I will make sure you're on the right track.

- (a) The energy of a black hole is $U = Mc^2$. Calculate the temperature of a Schwarzschild black hole with mass M .

Answer: Putting things together,

$$S = \frac{4\pi kG}{\hbar c} M^2 = \frac{4\pi kG}{\hbar c^5} U^2$$

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{8\pi kG}{\hbar c^5} U = \frac{8\pi kG}{\hbar c^3} M \quad \Rightarrow \quad T = \frac{\hbar c^3}{8\pi kG} \frac{1}{M}$$

Note that the smaller the black hole, the higher the temperature.

- (b) A black hole is a perfect blackbody, which will radiate energy due to its temperature. Calculate the rate of evaporation of a Schwarzschild black hole of mass M . Assume that only photons will be radiated. Note that the Stefan-Boltzmann constant $\sigma = \pi^2 k^4 / 60 \hbar^3 c^2$.

Answer: The power radiated by a blackbody is σeAT^4 . For a perfect blackbody $e = 1$. Since there's no compression-expansion work in outer space, $dU = dQ$. Therefore

$$\frac{dU}{dt} = c^2 \frac{dM}{dt} = -\sigma AT^4$$

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The negative sign is because the energy/mass of the black hole will *decrease* with the evaporation. Putting in everything,

$$\frac{dM}{dt} = -\frac{\hbar c^4}{2^{10} 15 \pi G^2} \frac{1}{M^2} = -r \frac{1}{M^2}$$

where I've called the constant up front r .

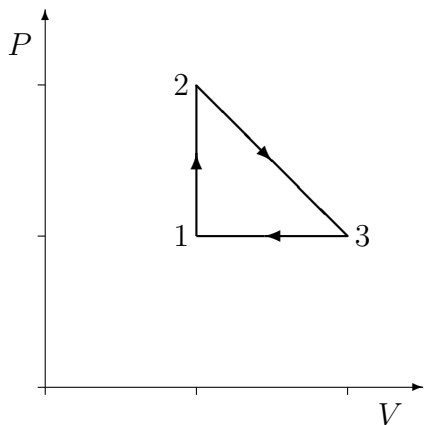
- (c) Calculate $t_{ev}(M)$, the time it takes for a Schwarzschild black hole to totally evaporate. Calculate this value for a typical stellar remnant black hole with $M = 20M_{\odot}$. (M_{\odot} is a solar mass.)

Answer: Rewrite the rate as $M^2 dM = -r dt$. We can integrate both sides, starting from $t = 0$ where the mass is M and going until t_{ev} where the mass is 0:

$$\int_M^0 dM M^2 = -r \int_0^{t_{ev}} dt \quad \Rightarrow \quad t_{ev} = \frac{M^3}{3r} = \frac{2^{10} 5 \pi G^2}{\hbar c^4} M^3$$

Putting in the numbers for $M = 20M_{\odot}$, we get $t_{ev} = 5.3 \times 10^{78}$ s, or 1.7×10^{71} years. A *lot* longer than the age of the universe!

- 2. (50 points)** You have a heat engine that uses a monatomic ideal gas as a working fluid, and that goes through the following cycle, where $P_2 = 2P_1$ and $V_3 = 2V_1$.



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- (a) Calculate the efficiency of the heat engine. Note, however, that you should not assume that the heat transfer to the engine during $2 \rightarrow 3$ is always the same sign. Find out if there are regions on the line segment where $Q < 0$, and if so, account for it as part of Q_h or Q_c accordingly.

Answer: For $1 \rightarrow 2$,

$$W_{12} = 0 \quad Q_{12} = \Delta U - W = C_v \Delta T = \frac{3}{2} (P_2 V_1 - P_1 V_1) = \frac{3}{2} P_1 V_1 > 0$$

For $3 \rightarrow 1$,

$$\begin{aligned} W_{31} &= -P \Delta V = P_1 V_1 \\ Q_{31} &= \Delta U - W = C_p \Delta T = \frac{5}{2} (P_1 V_1 - P_1 V_3) = -\frac{5}{2} P_1 V_1 < 0 \end{aligned}$$

For $2 \rightarrow 3$, the equation of the path is

$$P(V) = P_1 \left(3 - \frac{V}{V_1} \right)$$

Therefore, since $PV = NkT$, on this path,

$$P_1 \left(3 - \frac{V}{V_1} \right) V = NkT \quad \Rightarrow \quad P_1 \left(3 - \frac{2V}{V_1} \right) dV = Nk dT$$

Now, since $dW = -pdV$ and $dQ = dU - dW$,

$$\begin{aligned} dQ &= \frac{3}{2} Nk dT + PdV = P_1 \left[\frac{3}{2} \left(3 - \frac{2V}{V_1} \right) + \left(3 - \frac{V}{V_1} \right) \right] dV \\ &= \frac{P_1}{2} \left[15 - 8 \frac{V}{V_1} \right] dV \end{aligned}$$

For V increasing from V_1 until $2V_1$, $dV > 0$, and this means that $dQ > 0$ for $V_1 < V < \frac{15}{8}V_1$, and $dQ < 0$ for $\frac{15}{8}V_1 < V < 2V_1$. The two line segments must be accounted for separately, as part of Q_h and as part of Q_c . The positive and negative parts are:

$$Q_+ = \frac{P_1}{2} \int_{V_1}^{\frac{15}{8}V_1} dV \left(15 - 8 \frac{V}{V_1} \right) = \frac{49}{32} P_1 V_1 = 1.53 P_1 V_1$$

$$Q_- = \frac{P_1}{2} \int_{\frac{15}{8}V_1}^{2V_1} dV \left(15 - 8 \frac{V}{V_1} \right) = -\frac{1}{32} P_1 V_1 = -0.03 P_1 V_1$$

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The work done from $2 \rightarrow 3$ is straightforward; $W_{23} = -\frac{3}{2}P_1V_1$.

Adding positive and negative heats up separately,

$$Q_h = Q_{12} + Q_+ = \frac{97}{32}P_1V_1 = 3.03 P_1V_1$$

$$Q_c = -Q_{31} - Q_- = \frac{81}{32}P_1V_1 = 2.53 P_1V_1$$

$$W = -W_{12} - W_{23} - W_{31} = \frac{1}{2}P_1V_1 = 0.50 P_1V_1$$

The efficiency is

$$e = \frac{W}{Q_h} = \frac{16}{97} = 0.165$$

- (b) Find the efficiency of a Carnot engine operating between the maximum and minimum temperatures on this cycle.

Answer: We can start by remembering that isotherms are hyperbolas, and that the hyperbolas move away from the axes as T increases. Therefore, from symmetry, the maximum temperature on the cycle is midpoint on the $2 \rightarrow 3$ part. This is at $P = \frac{3}{2}P_1$ and $V = \frac{3}{2}V_1$, so that

$$T_h = \frac{9P_1V_1}{4Nk}$$

The minimum temperature is at point 1, so $T_c = P_1V_1/Nk$. The maximum efficiency is therefore

$$e_{\max} = 1 - \frac{T_c}{T_h} = \frac{5}{9} = 0.556$$