Homework Solutions 4 (Ryden Chapter 5)

5 Since energy density goes like $\varepsilon = \varepsilon_0\, a^{-3(1+w)}$, the scale factor for when $\varepsilon_m = \varepsilon_p$ is when $\varepsilon_{m,0}\, a^{-3} = \varepsilon_{p,0}\, a^{-3(1+w_p)}$, or

$$a_{mp} = \left(\frac{\varepsilon_{p,0}}{\varepsilon_{m,0}}\right)^{1/3w_p} = \left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/3w_p}$$

The Friedmann equation in this case is

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_{m,0}}{a^3} + \frac{1-\Omega_{m,0}}{a^{3(1+w_p)}}$$

When $a \gg a_{mp}$, the first term becomes negligible compared to the second, leaving, after a square root,

$$\dot{a} = H_0\sqrt{1-\Omega_{m,0}}\, a^{1-3(1+w_p)/2} \quad \Rightarrow \quad da\, a^{(1+3w_p)/2} = dt\, H_0\sqrt{1-\Omega_{m,0}}$$

Integrating this from now to the rip time means going from $a(t_0) = 1$ to $a(t_{\text{rip}}) = \infty$ in the a-integral, and t_0 to t_{rip} in the t-integral:

$$\int_1^\infty da\, a^{(1+3w_p)/2} = \int_{t_0}^{t_{\text{rip}}} dt\, H_0\sqrt{1-\Omega_{m,0}}$$

$$-\frac{1}{\frac{3}{2}(1+w_p)} = H_0\sqrt{1-\Omega_{m,0}}(t_{\text{rip}} - t_0)$$

Remembering that $w_p < -1$, we can rearrange this to

$$H_0\,(t_{\text{rip}} - t_0) = \frac{2}{3|1+w_p|}\,(1-\Omega_{m,0})^{-1/2}$$

With the numbers given,

$$t_{\text{rip}} - t_0 = 3.63 \times 10^{16}\ \text{s} = 1.15 \times 10^{11}\ \text{years}$$

6 We want $\ddot{a} = \dot{a} = 0$. The acceleration equation gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\,(\varepsilon_m + \varepsilon_q + 3w_q\varepsilon_q) = 0$$

This means $\varepsilon_m = -(1+3w_q)\varepsilon_q$.

For $\dot{a} = 0$, we can use the Friedmann equation so that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\varepsilon_m + \varepsilon_q\right) - \kappa\frac{c^2}{R_0^2 a^2} = 0$$

Now, $\varepsilon_m + \varepsilon_q = [1 - 1/(1+3w_q)]\varepsilon_m$. Since $w_q < -\frac{1}{3}$, $\varepsilon_m + \varepsilon_q > 0$ and therefore the first term in the Friedmann equation is positive. As a result, the second term must be negative, which requires that $\kappa = +1$.

With $a = 1$ at the present, the radius of curvature R_0 can also be found from the Friedmann equation:

$$\frac{8\pi G}{3c^2}\left[-(1+3w_q)+1\right]\varepsilon_q = \frac{c^2}{R_0^2} \qquad \Rightarrow \qquad R_0 = \frac{c^2}{\sqrt{-8\pi G w_q \varepsilon_q}}$$

8 Start with the Friedmann equation again, setting $\dot{a} = 0$ at the Big Bounce:

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \Omega_0 - \frac{\Omega_0 - 1}{a^2} \qquad \Rightarrow \qquad a_{\text{bounce}} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2}$$

When $\dot{a} \neq 0$, we can solve the Friedmann equation:

$$\frac{1}{H_0^2}\dot{a}^2 = \Omega_0 a^2 - \Omega_0 - 1 \qquad \Rightarrow \qquad \dot{a} = H_0\sqrt{\Omega_0}\sqrt{a^2 - a_{\text{bounce}}^2}$$

Integrating,

$$\int_{a_{\text{bounce}}}^{a} \frac{da'}{\sqrt{a'^2 - a_{\text{bounce}}^2}} = H_0\sqrt{\Omega_0}\int_{t_{\text{bounce}}}^{t} dt'$$

$$\cosh^{-1}\left(\frac{a}{a_{\text{bounce}}}\right) = H_0\sqrt{\Omega_0}(t - t_{\text{bounce}})$$

This gives us what we want:

$$a = a_{\text{bounce}}\cosh\left[H_0\sqrt{\Omega_0}(t - t_{\text{bounce}})\right]$$

The time elapsed is then, with $t = t_0$ and $a_0 = 1$,

$$t_0 - t_{\text{bounce}} = \frac{1}{H_0\sqrt{\Omega_0}}\cosh^{-1}\left(\sqrt{\frac{\Omega_0}{\Omega_0 - 1}}\right)$$