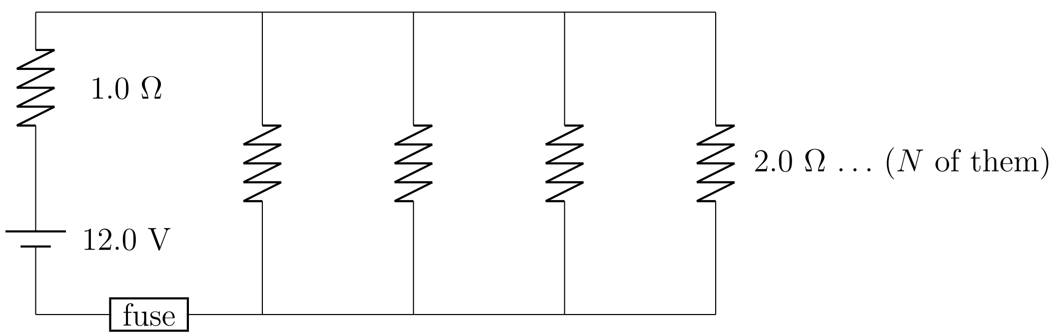
Solutions to Exam 2; Phys 186

1. (70 points) You have a circuit with a fuse in it to limit the current drawn from the battery, which supplies power to N 2.0 Ω resistors hooked in parallel. The 1.0 Ω resistor represents the internal resistance of the battery. The fuse acts like an ideal wire until a maximum current goes through it, at which point it burns up and breaks the circuit.



(a) Write down the junction, loop, and resistor equations for the circuit. Note that this is not as complicated as it looks, since many of your equations will be simplified to become identical. Ask me what I mean if you're confused.

Answer: Call the current from the battery I_0 and the currents through the resistors I_1 through I_N. There is a single junction equation, when the battery current splits into N parts:

$$I_0 = I_1 + I_2 + \ldots + I_N$$

The other junction gives the same equation, since all the split currents rejoin.

The resistor equations are $V = RI$ for each; it's easiest to directly use these in the loop equations. There are N independent loops in the circuit. The first loop equation is

$$12 \text{ V} = (1\,\Omega)I_0 + (2\,\Omega)I_1$$

The $N - 1$ other loop equations are

$$(2\,\Omega)I_1 = (2\,\Omega)I_2, \quad (2\,\Omega)I_2 = (2\,\Omega)I_3, \quad \ldots \quad (2\,\Omega)I_{N-1} = (2\,\Omega)I_N$$

All this means is that the split currents are all equal:

$$I_1 = I_2 = \ldots = I_N$$

So we end up with two equations

$$I_0 = N\, I_1, \qquad 12 \text{ A} = I_0 + 2\, I_1$$

and three unknowns, I_0, I_1, and N.

(b) If the fuse blows when the current through it is more than 10.0 A, what is the maximum number N_{max} of 2.0 Ω resistors that can be hooked up in this circuit?

Answer: We just need to solve our equations for when $I_0 = 10$ A. In that case, $I_1 = 10/N_{\text{max}}$ A, and we are left with

$$12 \text{ A} = 10 \text{ A} + 2\, \frac{10}{N_{\text{max}}} \text{ A}$$

which means that $N_{\text{max}} = 10$.

(c) Could you increase N_{max} by hooking up a capacitor in series with the 1.0 Ω resistor? In parallel? Explain.

Answer: If you hooked up a capacitor in series, it would charge up, and once it was fully charged, no more charges would be moving on that wire: the current I_0 would become zero. That would not help.

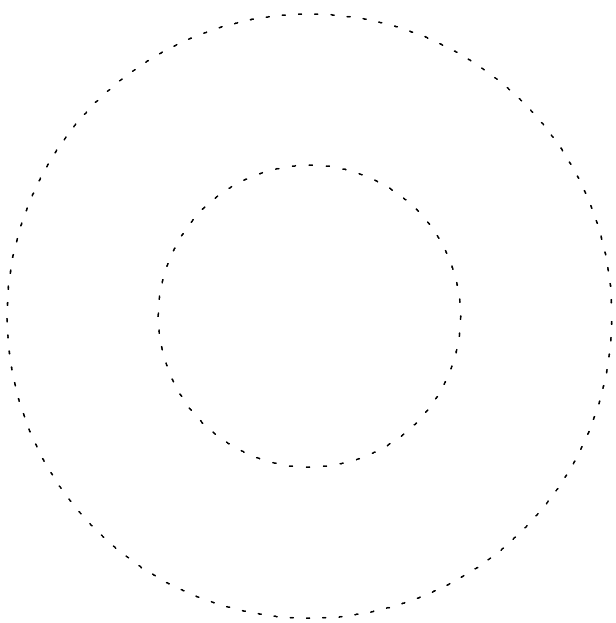
In parallel, again, the capacitor would charge up, and no current would go through afterwards. But that would just leave us with an equivalent of the original circuit after a while, so N_{max} would not be affected.

In other words, a capacitor would not do the job, however it is connected.

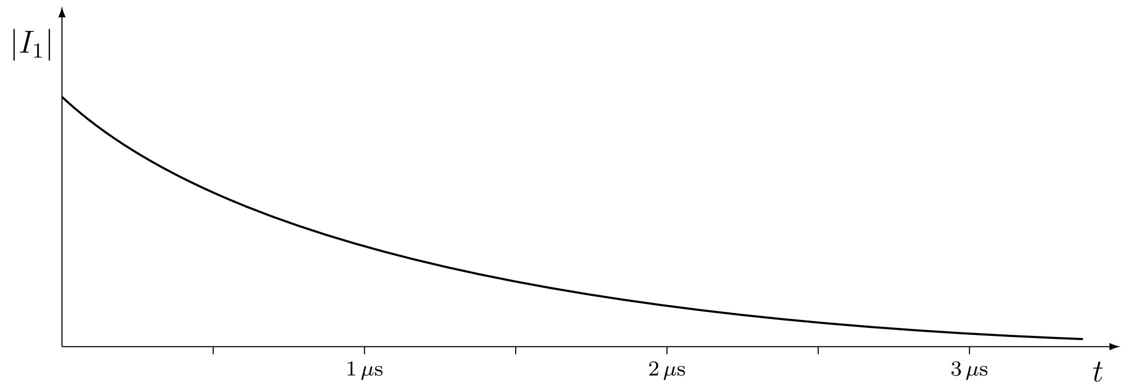
(d) What does this problem tell you about the consequences of hooking up lots of appliances to a single wall socket by using extension cords?

Answer: If you keep adding appliances, your fuse or circuit breaker will have to break the circuit so you don't draw too much current and start a fire.

2. (30 points) You have an open switch, a battery that supplies a voltage of 5.0 V, a capacitor with $C = 0.44\,\mu\text{F}$, and a resistor $R_1 = 2.5\,\Omega$. On the outer circle below, which has a radius of 0.40 m, arrange these four circuit elements in such a way that you have a single current through everything, and the circuit is circular: fill in the arcs between with non-dotted lines representing ideal wires. Then, on the inner circle, place a resistor with $R_2 = 1.2\,\Omega$, where this resistor is the only element in another circular circuit with the same center and radius 0.20 m.



(a) At time $t = 0$, you close the switch. Draw a qualitative graph of the current $|I_1|$ in the outer circle. Calculate exact values for $t = 0$, $t = 1\,\mu\text{s}$, and $t = 2\,\mu\text{s}$.



Answer: Your circuit will be a circuit that charges up a capacitor. At $t = 0$, the voltage across the capacitor will be zero, and the battery voltage will be fall fully across the resistor. The starting current, then, will be $I_1 = V_1/R_1 = 5\,\mathrm{V}/2.5\,\Omega = 2\,\mathrm{A}$.

As time passes, the current will show the characteristic exponential decay, with $I_1 = (V_1/R_1)\,e^{-t/\tau}$, where the time scale $\tau = R_1 C_1 = (0.44\,\mu\mathrm{s})(2.5\,\Omega) = 1.1\,\mu\mathrm{s}$. Doing the calculations,

$$I_1(1\,\mu\mathrm{s}) = (2\,\mathrm{A})\,e^{-1/1.1} = 0.81\,\mathrm{A}, \qquad I_1(2\,\mu\mathrm{s}) = (2\,\mathrm{A})\,e^{-2/1.1} = 0.32\,\mathrm{A}$$

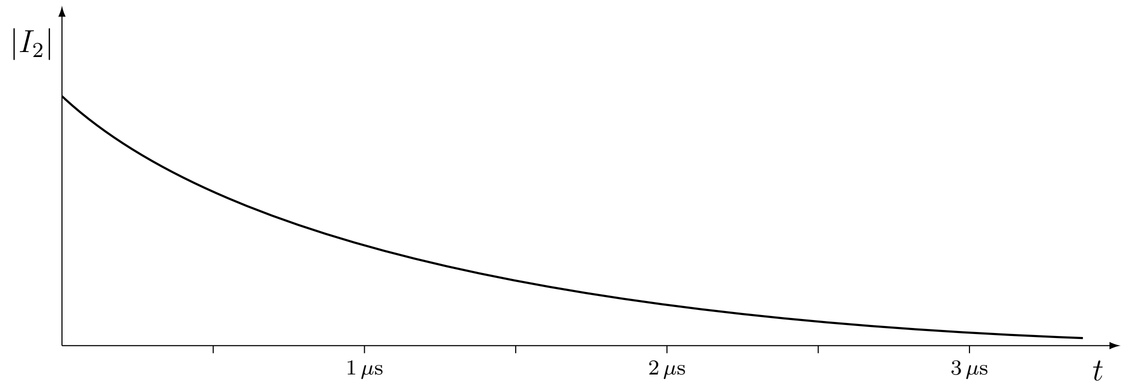
The graph is, naturally, an exponential decay curve.

(b) Indicate, on the diagram with the circles, the direction of the magnetic field produced by I_1, both inside and outside of the outer circuit.

Answer: The direction will be according to the right-hand rule, and will depend on the current direction as determined by how you draw the battery. If the current I_1 is clockwise, the magnetic field inside will be into the page.

(c) Now draw a graph of $|I_2|$, the current in the inner circle. Make an *estimate* of this current at $t = 0$. Use the following math for a rate of change you may need:

$$\frac{d}{dt}\,e^{-t/\tau} = -\frac{1}{\tau}\,e^{-t/\tau}$$



Answer: We can estimate the magnetic field inside the smaller circuit by using the magnetic field magnitude at the center of a circular current. This is $B = \mu_0 I/2r$, an option that you did not use in your Assignment 5.1(b). Therefore, the induced current is

$$\begin{aligned} |I_2| &= \frac{1}{R_2}\left|\frac{d}{dt}\Phi\right| = \frac{1}{R_2}\left|\frac{d}{dt}\left(\frac{\mu_0 I_1}{2r_1}\pi r_2^2\right)\right| = \frac{V_1}{R_1 R_2}\frac{\pi\mu_0 r_2^2}{2r_1}\left|\frac{d}{dt}\,e^{-t/\tau}\right| \\ &= \frac{V_1}{R_1^2 R_2 C_1}\frac{\pi\mu_0 r_2^2}{2r_1}\,e^{-t/R_1C_1} = (0.30\,\mathrm{A})\,e^{-t/R_1C_1} \end{aligned}$$

At $t = 0$, this current is $|I_2| = 0.30\,\mathrm{A}$. Again, the graph will be an exponential decay, with the same time scale τ as before.

3. (30 points) Say you're doing the lab where you accelerated and shot a beam of electrons onto a screen. The mass of an electron is $m_e = 511\,\mathrm{keV}/c^2$.

(a) You accelerated the electrons through a voltage difference of up to 5.00 kV on your dial. At $V_a = 5.00\,\mathrm{kV}$, then, what is the kinetic energy of the electrons in the beam, in units of keV? *Hint:* 1 eV is literally the electron charge magnitude e multiplied by 1 V. Therefore, you shouldn't need any real calculation to get this answer.

Answer: $e(5\,\mathrm{kV}) = 5\,\mathrm{keV}$.

(b) What fraction of the speed of light are these electrons traveling? Use $\frac{1}{2}m_e v^2$ for your kinetic energy, as in your homework and the lab.

Answer: The usual energy conservation gives

$$5\,\text{keV} = \frac{1}{2}(511\,\text{keV})\frac{v^2}{c^2} \quad \Rightarrow \quad \frac{v}{c} = \sqrt{\frac{2\cdot 5}{511}} = 0.140$$

14% of the speed of light. I mentioned that our electrons in the lab were fast.

(c) Recalculate the fraction of the speed of light the electron has, using a more appropriate expression for kinetic energy.

Answer: A more accurate calculation of the speed would go like this. The total energy of the electron is $\gamma m_e c^2$, while the energy of an electron at rest, where $\gamma = 1$, is $m_e c^2$. Therefore, the kinetic energy, which is the additional energy an object has due its motion, must be the difference: $K = (\gamma - 1)m_e c^2$. We can use this relativistic form of kinetic energy to recalculate the fraction of the speed of light the electron has.

Here, γ depends on v/c, so after some algebra,

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\left(1 + \frac{K}{m_e c^2}\right)^2}} = 0.139$$

Again, 14% of the speed of light.

(d) Compare your results in (b) and (c). Do you think relativity was important enough to account for in your lab?

Answer: The answers are very similar; you did not need relativity. While very fast, the electrons are not close enough to the speed of light for relativistic effects to become important.

(e) Say you got a lot more expensive equipment that could provide an accelerating voltage of up to $V_a = 500\,\text{kV}$. In that case, what would you calculate the speed of the electrons to be (as a fraction of the speed of light) if you used $K = \frac{1}{2}m_e v^2$?

Answer: Same calculation:

$$500\,\text{keV} = \frac{1}{2}(511\,\text{keV})\frac{v^2}{c^2} \quad\Rightarrow\quad \frac{v}{c} = \sqrt{\frac{2\cdot 500}{511}} = 1.40$$

140% of the speed of light. That should raise an eyebrow.

(f) Redo the calculation for v as a fraction of the speed of light with $V_a = 500\,\text{kV}$, but now using the correct expression for kinetic energy.

Answer: Same calculation:

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\left(1 + \frac{K}{m_e c^2}\right)^2}} = 0.863$$

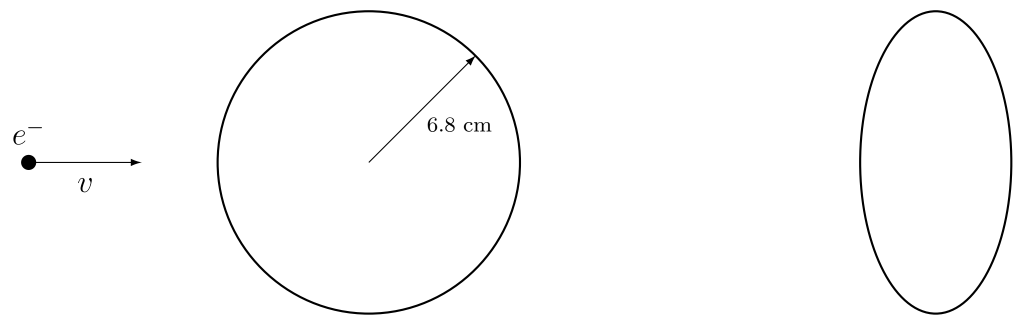
86% of the speed of light.

(g) Compare your results with different expressions for kinetic energy in (e) and (f) and interpret what they mean.

Answer: Clearly the nonrelativistic calculation is wrong: faster than light? The electrons are now moving close enough to the speed of light that relativity becomes important, and the proper relativistic kinetic energy expression is necessary.

(h) Again, $V_a = 500\,\text{kV}$. In the lab reference frame, the copper coils with the current providing the magnetic field were circles with a radius of about $R = 6.8\,\text{cm}$. Sketch how the coils look in the electrons' reference frame, and calculate the appropriate dimensions (height, width) for the coil in that frame.

Answer: Here, $\gamma = 1.98$. The coil is not moving in the lab frame, therefore its dimensions are proper lengths. Length contraction will occur along the direction of motion, so the width will contract down to $2 \cdot 6.8/1.98 = 6.9\,\text{cm}$. The height is perpendicular to the direction of motion, so this will not be contracted, remaining at $2 \cdot 6.8 = 13.6\,\text{cm}$.



lab frame electron frame