Homework Solutions 5 (Ryden Chapter 6 & 7)

6.5 Combining equations 6.27 for f and 6.34 for angular size $\delta\theta$,

$$\Sigma \propto \frac{f}{(\delta\theta)^2} \propto \frac{L}{S_\kappa(r)^2(1+z)^2} \frac{S_\kappa(r)^2}{l^2(1+z)^2} \propto \frac{L}{l^2(1+z)^4}$$

Since the $S_\kappa(r)$-dependence cancels out, this result is independent of curvature, and is indeed correct for any universe with a Robertson-Walker metric. However, it's also independent of the acceleration of the expansion of the universe, so it has no information about q_0.

6.6 From the relationship between t_0 and t_e,

$$\delta t_e = \frac{\delta t_0}{1+z} = \tfrac{1}{2}\ \text{day} = 4.32 \times 10^4\ \text{s}$$

Therefore

$$R_{\max} = c\,\delta t_e = 1.3 \times 10^{13}\ \text{m} = 4.2 \times 10^{-4}\ \text{pc}$$

Using figure 6.4 with $z = 5$, $d_A \approx 0.3\,c/H_0 = 1.3 \times 10^9$ pc. And the angular size

$$\delta\theta = \frac{R_{\max}}{d_A} = 3.2 \times 10^{-13}\ \text{rad}$$

Thats should be too small for astronomers to observe.

6.8 Combining equations 5.50, 6.21, and 6.29, we get f for a standard candle,

$$f = \frac{H_0^2 L (1+3w)^2}{16\pi c^2 (1+z)^2 \left[1 - (1+z)^{-(1+3w)/2}\right]^2}$$

When $w = -\frac{1}{3}$, the integral in equation 5.49 gives a log: $d_p(t_0) = \frac{c}{H_0}\ln(1+z)$, resulting in

$$f = \frac{H_0^2 L}{4\pi c^2 \left[(1+z)\ln(1+z)\right]^2}$$

Now, r is the proper distance to the standard candle, once again given by equation 5.50. The number of standard candles r to $r+dr$ away is the density n_0 times the volume of the spherical shell of radius r:

$$dN = \left(4\pi r^2\, dr\right) n_0$$

We are looking for those in the range z to $z + dz$. So we use $dr = \frac{dr}{dz}dz$, taking the derivative of equation 5.50. We get

$$dN = \frac{4\pi c n_0}{H_0} d_p(t_0)^2 (1+z)^{-3(1+w)/2}\, dz$$

The intensity is the flux f for each standard candle divided by the full solid angle of 4π steradians, so

$$dJ = \frac{f}{4\pi} dN = \frac{n_0 L c}{4\pi H_0}(1+z)^{-(7+3w)/2}\, dz$$

We integrate this over all z to get the intensity:

$$J = \int_{z=0}^{z=\infty} dJ = \frac{n_0 L c}{2\pi H_0 (5+3w)}$$

As in your exam problem, the horizon distance can diverge when $w < -\frac{1}{3}$, since

$$d_{\text{hor}} = \int_0^\infty dz\, \frac{dr}{dz} \propto (1+z)^{-(1+3w)/2}\Big|_0^\infty$$

But the J integral is different, diverging when $w < -\frac{5}{3}$. In the range of $-\frac{5}{3} < w < -\frac{1}{3}$, the standard candles at large distances are so far redshifted that even the light seen from an infinity of them gives a finite brightness.

7.4 Equation 7.2 gives $M(r) = v^2 r/G$ for a spherically symmetric halo. The density as a function of distance is the mass dM at radius r divided by volume of a spherical shell at radius r, $4\pi r^2\, dr$:

$$\rho(r) = \frac{1}{4\pi r^2}\frac{dM}{dr} = \frac{v^2}{4\pi G r^2}$$

Mining the book for the relevant numbers, we get $v = 235$ km/s from page 130, which gives

$$\rho(r) = (1.03 \times 10^{-21}\ \text{kg/m}^3)\left(\frac{r}{8.2\ \text{kpc}}\right)^2$$

using peculiar units like equation 7.12, where 8.2 kpc is the radius of the Sun's orbit around the galactic center.

The cosmological constant gives a density $\rho_\Lambda = 0.7\rho_{c,0}$, which at $r = 8.2$ kpc is about 6×10^{-6} of the dark matter density $\rho(r)$. Much farther, at

$r \approx 100$ kpc, the dark energy density is still only about 10^{-3} of $\rho(r)$. So while Λ is important at cosmological scales, it shouldn't affect local dynamics.

7.6 Consider a spherical shell of thickness dr at a radius of r from the sun. Neutrinos travel at very nearly c and interact so rarely that we can neglect any absorption. The time it takes a neutrino to cross dr is dr/c. The number of solar neutrinos inside that shell, then, is $dN = dr\,(2 \times 10^{38}/\mathrm{s})/c$. The number density is

$$n = \frac{dN}{dV} = \frac{(2 \times 10^{38}/\mathrm{s})\,dr}{c\,4\pi r^2\,dr}$$

At a distance $r = 1$ AU, this gives $n = 2.4 \times 10^6/\mathrm{m}^3$.

Say a typical human is around 60 kg, and her density is that of water. Their volume is then 0.06 m^3. The number of solar neutrinos in that volume is

$$N_s = nV = 1.5 \times 10^5$$

The cosmic neutrino density is in equation 7.49; this gives a number of $N_c = 2.0 \times 10^7$, a lot larger.