6.5 Combining equations 6.27 for \( f \) and 6.34 for angular size \( \delta \theta \),

\[
\Sigma \propto \frac{f}{(\delta \theta)^2} \propto \frac{L}{S_\kappa(r)^2(1+z)^2} \frac{S_\kappa(r)^2}{l^2(1+z)^2} \propto \frac{L}{l^2(1+z)^4}
\]

Since the \( S_\kappa(r) \)-dependence cancels out, this result is independent of curvature, and is indeed correct for any universe with a Robertson-Walker metric. However, it’s also independent of the acceleration of the expansion of the universe, so it has no information about \( q_0 \).

6.6 From the relationship between \( t_0 \) and \( t_e \),

\[
\delta t_e = \frac{\delta t_0}{1+z} = \frac{1}{2} \text{ day} = 4.32 \times 10^4 \text{ s}
\]

Therefore

\[
R_{\text{max}} = c \delta t_e = 1.3 \times 10^{13} \text{ m} = 4.2 \times 10^{-4} \text{ pc}
\]

Using figure 6.4 with \( z = 5 \), \( d_A \approx 0.3 \frac{c}{H_0} = 1.3 \times 10^9 \text{ pc} \). And the angular size

\[
\delta \theta = \frac{R_{\text{max}}}{d_A} = 3.2 \times 10^{-13} \text{ rad}
\]

Thats should be too small for astronomers to observe.

6.8 Combining equations 5.50, 6.21, and 6.29, we get \( f \) for a standard candle,

\[
f = \frac{H_0^2 L (1 + 3w)^2}{16 \pi c^2 (1 + z)^2 \left[ 1 - (1 + z)^{-(1+3w)/2} \right]^2}
\]

When \( w = -\frac{1}{3} \), the integral in equation 5.49 gives a log: \( d_p(t_0) = \frac{c}{H_0} \ln(1+z) \), resulting in

\[
f = \frac{H_0^2 L}{4 \pi c^2 \left[ (1 + z) \ln(1+z) \right]^2}
\]

Now, \( r \) is the proper distance to the standard candle, once again given by equation 5.50. The number of standard candles \( r \) to \( r+dr \) away is the density \( n_0 \) times the volume of the spherical shell of radius \( r \):

\[
dN = \left( 4 \pi r^2 dr \right) n_0
\]
We are looking for those in the range $z$ to $z + dz$. So we use $dr = \frac{dr}{dz} dz$, taking the derivative of equation 5.50. We get

$$dN = \frac{4\pi cn_0}{H_0} d_p(t_0)^2 (1 + z)^{-3(1+w)/2} dz$$

The intensity is the flux $f$ for each standard candle divided by the full solid angle of $4\pi$ steradians, so

$$dJ = f \frac{4\pi}{4\pi} dN = \frac{n_0 Lc}{4\pi H_0} (1 + z)^{-(7+3w)/2} dz$$

We integrate this over all $z$ to get the intensity:

$$J = \int_{z=0}^{z=\infty} dJ = \frac{n_0 Lc}{2\pi H_0 (5 + 3w)}$$

As in your exam problem, the horizon distance can diverge when $w < -\frac{1}{3}$, since

$$d_{\text{hor}} = \int_0^{\infty} dz \frac{dr}{dz} \propto (1 + z)^{-(1+3w)/2} \bigg|_0^\infty$$

But the $J$ integral is different, diverging when $w < -\frac{5}{3}$. In the range of $-\frac{5}{3} < w < -\frac{1}{3}$, the standard candles at large distances are so far redshifted that even the light seen from an infinity of them gives a finite brightness.

7.4 Equation 7.2 gives $M(r) = v^2 r/G$ for a spherically symmetric halo. The density as a function of distance is the mass $dM$ at radius $r$ divided by volume of a spherical shell at radius $r$, $4\pi r^2 dr$:

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM}{dr} = \frac{v^2}{4\pi G r^2}$$

Mining the book for the relevant numbers, we get $v = 235$ km/s from page 130, which gives

$$\rho(r) = (1.03 \times 10^{-21} \text{ kg/m}^3) \left( \frac{r}{8.2 \text{ kpc}} \right)^2$$

using peculiar units like equation 7.12, where 8.2 kpc is the radius of the Sun’s orbit around the galactic center.

The cosmological constant gives a density $\rho_\Lambda = 0.7 \rho_{c,0}$, which at $r = 8.2$ kpc is about $6 \times 10^{-6}$ of the dark matter density $\rho(r)$. Much farther, at
$r \approx 100$ kpc, the dark energy density is still only about $10^{-3}$ of $\rho(r)$. So while $\Lambda$ is important at cosmological scales, it shouldn’t affect local dynamics.

### 7.6

Consider a spherical shell of thickness $dr$ at a radius of $r$ from the sun. Neutrinos travel at very nearly $c$ and interact so rarely that we can neglect any absorption. The time it takes a neutrino to cross $dr$ is $dr/c$. The number of solar neutrinos inside that shell, then, is $dN = dr \left(2 \times 10^{38}/s\right)/c$. The number density is

$$n = \frac{dN}{dV} = \frac{(2 \times 10^{38}/s) \, dr}{c \pi r^2 \, dr}$$

At a distance $r = 1$ AU, this gives $n = 2.4 \times 10^6/m^3$.

Say a typical human is around 60 kg, and her density is that of water. Their volume is then 0.06 m$^3$. The number of solar neutrinos in that volume is

$$N_s = nV = 1.5 \times 10^5$$

The cosmic neutrino density is in equation 7.49; this gives a number of $N_c = 2.0 \times 10^7$, a lot larger.