Homework Solutions #2 (McIntyre Chapter 2)

7 First the eigenvalues:

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta = \lambda^2 - 1 = 0$$

The eigenvalues of S_n are therefore $\pm\frac{\hbar}{2}$. The corresponding *non*-normalized eigenvectors are, if you solve for them straightforwardly,

$$\begin{pmatrix} \sin\theta \\ (1-\cos\theta)e^{i\phi} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sin\theta \\ (-1-\cos\theta)e^{i\phi} \end{pmatrix}$$

To get rid of the nuisances of the ± 1, use $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and also $\cos\theta = 1-2\sin^2\frac{\theta}{2}$ for the $+\frac{\hbar}{2}$ eigenvalue and $\cos\theta = 2\cos^2\frac{\theta}{2}-1$ for the $-\frac{\hbar}{2}$ eigenvalue. After normalization, we end up with

$$|+\rangle_n \doteq \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \qquad |-\rangle_n \doteq \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

9 The relevant expectation values are just the ++ matrix elements of the relevant operators in the $\{|+\rangle, |-\rangle\}$ basis. These are all known:

$$\langle +|S_x|+\rangle = 0 \quad \langle +|S_x^2|+\rangle = \frac{\hbar^2}{4} \qquad \Delta S_x = \left(\langle +|S_x^2|+\rangle - \langle +|S_x|+\rangle^2\right)^{\frac{1}{2}} = \frac{\hbar}{2}$$

$$\langle +|S_y|+\rangle = 0 \quad \langle +|S_y^2|+\rangle = \frac{\hbar^2}{4} \qquad \Delta S_y = \left(\langle +|S_y^2|+\rangle - \langle +|S_y|+\rangle^2\right)^{\frac{1}{2}} = \frac{\hbar}{2}$$

$$\langle +|S_z|+\rangle = \frac{\hbar}{2} \quad \langle +|S_z^2|+\rangle = \frac{\hbar^2}{4} \qquad \Delta S_z = \left(\langle +|S_z^2|+\rangle - \langle +|S_z|+\rangle^2\right)^{\frac{1}{2}} = 0$$

17 The vector components are given in the basis of the S_z eigenstates

(a) We can just read off the magnitude squares of the components:

$$\mathcal{P}_{S_z=\hbar} = \left|\frac{1}{\sqrt{30}}\right|^2 = \frac{1}{30} \quad \mathcal{P}_{S_z=0} = \left|\frac{2}{\sqrt{30}}\right|^2 = \frac{4}{30} \quad \mathcal{P}_{S_z=-\hbar} = \left|\frac{5i}{\sqrt{30}}\right|^2 = \frac{25}{30}$$

Therefore the expectation value is

$$\langle S_z\rangle = \frac{1}{30}(\hbar) + \frac{4}{30}(0) + \frac{25}{30}(-\hbar) = -0.8\hbar$$

(b)

$$\langle\psi|S_x|\psi\rangle = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 & 2 & -5i \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} = \frac{\hbar\sqrt{2}}{15}$$

23 Matrices, matrices...

(a) Unless $a_2 = a_3$, these won't commute:

$$[A, B] = AB - BA \doteq \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_2 b_2 \\ 0 & a_3 b_2 & 0 \end{pmatrix} - \begin{pmatrix} b_1 a_1 & 0 & 0 \\ 0 & 0 & b_2 a_3 \\ 0 & b_2 a_2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_2(a_2 - a_3) \\ 0 & b_2(a_3 - a_2) & 0 \end{pmatrix} \neq 0$$

(b) A is already diagonal, therefore its eigenvectors are $|1\rangle$, $|2\rangle$, and $|3\rangle$, with eigenvalues a_1, a_2. and a_3. For B,

$$\begin{vmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{vmatrix} = (b_1 - \lambda)(\lambda^2 - b_2^2) = 0 \quad \Rightarrow \quad \lambda = b_1, b_2, -b_2$$

The eigenvectors are

$$|b_1\rangle = |1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |b_2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad |-b_2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(c)

$$|2\rangle = \frac{1}{\sqrt{2}}|b_2\rangle + \frac{1}{\sqrt{2}}|-b_2\rangle \qquad \Rightarrow \qquad \mathcal{P}_{B=b_2} = \mathcal{P}_{B=-b_2} = \frac{1}{2}$$

After the measurement, the system will be left in either state $|b_2\rangle$ or $|-b_2\rangle$. In either case, we can read the probabilities off the squares of the components:

$$\mathcal{P}_{A=a_2} = \mathcal{P}_{A=a_3} = \frac{1}{2}$$

(d) Since A and B don't commute, they are not compatible observables. Obtaining information about one destroys information about the other.