Homework Solutions #4 (McIntyre Chapter 5)

2 This is straightforward.

(a) $\langle\psi|\psi\rangle = A^*A(1+1+1) = 1$. Choose $A = \frac{1}{\sqrt{3}}$.

(b) E_1, E_2, and E_3 are possible, with equal probabilities of $\frac{1}{3}$.

(c) $\langle E\rangle = \frac{1}{3}(E_1 + E_2 + E_3) = \frac{14}{3}E_1$

(d) $|\psi(t)\rangle = A(e^{-i\omega_1 t}|\varphi_1\rangle - e^{-4i\omega_1 t}|\varphi_2\rangle + ie^{-9i\omega_1 t}|\varphi_3\rangle)$, where $\omega_1 = E_1/\hbar$.

(e) The same as (b), since H does not depend on t.

5 Obviously $\langle x\rangle = \frac{L}{2}$, but let's calculate everything.

$$\langle x\rangle = \langle\varphi_n|x|\varphi_n\rangle = \frac{2}{L}\int_0^L dx\, x\, \sin^2\left(\frac{n\pi}{L}x\right) = \frac{L}{2}$$

$$\langle x^2\rangle = \langle\varphi_n|x^2|\varphi_n\rangle = \frac{2}{L}\int_0^L dx\, x^2\, \sin^2\left(\frac{n\pi}{L}x\right) = \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)L^2$$

$$\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{\frac{1}{2}} = \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}\, L$$

$$\langle p\rangle = \langle\varphi_n|p|\varphi_n\rangle = -i\hbar\frac{2}{L}\int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\frac{\partial}{\partial x}\sin\left(\frac{n\pi}{L}x\right) =$$

$$-i\hbar\frac{2n\pi}{L^2}\int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right) = 0$$

$$\langle p^2\rangle = 2m\langle E\rangle = 2m\langle\varphi_n|H|\varphi_n\rangle = 2mE_n = \frac{n^2\pi^2\hbar^2}{L^2}$$

$$\Delta p = \left(\langle p^2\rangle - \langle p\rangle^2\right)^{\frac{1}{2}} = \frac{n\pi\hbar}{L}$$

8 Normalization means

$$\int_0^L dx\, \psi^*\psi = |A|^2\int_0^L dx\, x^2(L-x)^2 = 1 \quad\Rightarrow\quad |A| = \sqrt{\frac{30}{L^5}}$$

For the time evolution, we need to find the components of the state in the energy basis.

$$\langle\varphi_n|\psi(0)\rangle = \sqrt{\frac{2}{L}}\int_0^L dx\,\sin\left(\frac{n\pi}{L}x\right)\,Ax(L-x) = \frac{4\sqrt{15}}{(n\pi)^3}\left[1-(-1)^n\right]$$

Since $\psi(x)$ is symmetric with respect to reflections about $x=\frac{L}{2}$, we would expect that all the even n components are zero, which is the case.

The time evolution is

$$\psi(x,t) = e^{-i\frac{t}{\hbar}\hat{H}}\psi(x,0) = e^{-i\frac{t}{\hbar}\hat{H}}\sum_n \psi_n\varphi_n(x) = \sum_n \psi_n e^{-in^2\omega_1 t}\varphi_n(x)$$

where $\psi_n = \langle\varphi_n|\psi(0)\rangle$ and $\omega_1 = E_1/\hbar$. This doesn't look like it has any easier functional form.

The expectation value $\langle x\rangle = \frac{L}{2}$ always. Consider the operator $\mathcal{Q}$, which effects a reflection of a function about $x=\frac{L}{2}$. (Clearly $\mathcal{Q}$ is closely related to parity.) $\psi(x,0)$ is a $\mathcal{Q}$ eigenstate, and $[H,\mathcal{Q}]=0$. Therefore $\psi(x,t)$ is also a $\mathcal{Q}$ eigenstate, and all $\mathcal{Q}$ eigenstates give $\langle x\rangle = \frac{L}{2}$.

But that might not be obvious, so here's a brute force calculation:

$$\langle x\rangle = \int_0^L dx\,\left(\sum_n \psi_n^* e^{in^2\omega_1 t}\varphi_n^*\right) x \left(\sum_m \psi_m e^{-im^2\omega_1 t}\varphi_m\right) =$$

$$\sum_{nm}\psi_n^*\psi_m e^{i(n^2-m^2)\omega_1 t}\int_0^L dx\,\varphi_n^*(x)x\varphi_m(x)$$

If you do the integral,

$$\int_0^L dx\,\varphi_n^*(x)x\varphi_m(x) = \frac{2}{L}\int_0^L dx\,x\,\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) = \frac{L}{2}\delta_{nm}$$

Putting this in,

$$\langle x\rangle = \frac{L}{2}\sum_n \psi_n^*\psi_n = \frac{L}{2}$$

since the sum above is the sum of all energy probabilities, which is normalized to 1.

17 The probabilities for $x>a$ and $x<-a$ are the same. Therefore

$$\mathcal{P}_{|x|>a} = 2|A|^2\int_a^\infty dx\,e^{-2qx} = \frac{|A|^2 e^{-2qa}}{q} = \frac{a|A|^2 e^{-2\sqrt{z_0^2-z^2}}}{\sqrt{z_0^2-z^2}}$$

We now need to normalize the ground state to find $|A|^2$.

$$\mathcal{P}_{|x|<a} = |D|^2 \int_{-a}^{a} dx \, \cos^2(kx) = a|D|^2 \left[\frac{\sin(2ka)}{2ka} + 1\right] = a|D|^2 \left[\frac{\sin(2z)}{2z} + 1\right]$$

Since the probabilities sum to 1,

$$\frac{a|A|^2 e^{-2\sqrt{z_0^2 - z^2}}}{\sqrt{z_0^2 - z^2}} + a|D|^2 \left[\frac{\sin(2z)}{2z} + 1\right] = 1$$

From the quantization conditions for even parity (the ground state is even), we also get $Ae^{-qa} = D\cos ka$ and $qAe^{-qa} = kD\sin ka$. Using these, we get, after some algebra not worth reproducing,

$$\mathcal{P}_{|x|>a} = \frac{z^2}{z_0^2\left(1 + \sqrt{z_0^2 - z^2}\right)}$$