Homework Solutions #10 (McIntyre Ch. 13)

3 Six symmetric combinations:

$$|{-1}{-1}\rangle \qquad |00\rangle \qquad |11\rangle$$

$$\frac{1}{\sqrt{2}}\left(|{-1}0\rangle + |0{-1}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|{-1}1\rangle + |1{-1}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

Three antisymmetric combinations:

$$\frac{1}{\sqrt{2}}\left(|{-1}0\rangle - |0{-1}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|{-1}1\rangle - |1{-1}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

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(a) Distinguishable particles:

$$\varphi = \varphi_1(x_1)\varphi_2(x_2) = \frac{2}{L}\sin\left(\frac{\pi}{L}x_1\right)\sin\left(\frac{2\pi}{L}x_2\right)$$

$$P(x_1,x_2)dx_1dx_2 = \frac{4}{L^2}\sin^2\left(\frac{\pi}{L}x_1\right)\sin^2\left(\frac{2\pi}{L}x_2\right)dx_1dx_2$$

Change variables to $u = x_1 + x_2$ and $v = x_1 - x_2$:

$$P(u,v)\,du\,dv = \frac{1}{2}\frac{4}{L^2}\sin^2\left(\frac{\pi}{2L}(u+v)\right)\sin^2\left(\frac{\pi}{L}(u-v)\right)\,du\,dv$$

We now need to integrate out u to get $P(v)\,dv = \int du\,P(u,v)\,dv$. From symmetry, it's clear that $P(-v) = P(v)$, so we concentrate only on $0 \le v \le L$. For fixed positive v, u varies between a minimum of v and a maximum of $2L - v$. Therefore

$$\begin{aligned} P(v)\,dv &= \frac{2}{L^2}\int_v^{2L-v} du\,\sin^2\left(\frac{\pi}{2L}(u+v)\right)\sin^2\left(\frac{\pi}{L}(u-v)\right)\,dv \\ &= \frac{1}{12\pi L}\left[12\left(\pi - \frac{\pi}{L}v\right) + 8\sin\left(\frac{2\pi}{L}v\right) - \sin\left(\frac{4\pi}{L}v\right)\right]\,dv \end{aligned}$$

To include $-L \le v < 0$, we write this as

$$P(v)\,dv = \frac{1}{12\pi L}\left[12\left(\pi - \frac{\pi}{L}|v|\right) + 8\sin\left(\frac{2\pi}{L}|v|\right) - \sin\left(\frac{4\pi}{L}|v|\right)\right]\,dv$$

(b) Symmetric spatial state:

$$\varphi_S = \frac{1}{\sqrt{2}}\left[\varphi_1(x_1)\varphi_2(x_2) + \varphi_1(x_2)\varphi_2(x_1)\right]$$

$$\begin{aligned} P_S(x_1,x_2)dx_1dx_2 &= \frac{1}{2}\left[\varphi_1^2(x_1)\varphi_2^2(x_2) + \varphi_1^2(x_2)\varphi_2^2(x_1)\right]dx_1dx_2 + \\ &\quad \varphi_1(x_1)\varphi_2(x_2)\varphi_1(x_2)\varphi_2(x_1)dx_1dx_2 \end{aligned}$$

From symmetry, the first term will give a $P(v)$ identical to that for indistinguishable particles. The new contribution from the exchange is due to the second term. Let us concentrate on that.

$$P_{ex}(x_1,x_2)dx_1dx_2 =$$
$$\frac{4}{L^2}\sin\left(\frac{\pi}{L}x_1\right)\sin\left(\frac{2\pi}{L}x_2\right)\sin\left(\frac{\pi}{L}x_2\right)\sin\left(\frac{2\pi}{L}x_1\right)dx_1dx_2$$

Changing variables and using a trigonometric identity,

$$P_{ex}(u,v)\,du\,dv =$$
$$\frac{1}{2}\frac{1}{L^2}\left[\cos\left(\frac{\pi}{L}v\right) - \cos\left(\frac{\pi}{L}u\right)\right]\left[\cos\left(\frac{2\pi}{L}v\right) - \cos\left(\frac{2\pi}{L}u\right)\right]du\,dv$$

Integrating out v for $0 \le v \le L$ as before,

$$P_{ex}(v)\,dv =$$
$$\frac{1}{2L^2}\int_v^{2L-v} du\left[\cos\left(\frac{\pi}{L}v\right) - \cos\left(\frac{\pi}{L}u\right)\right]\left[\cos\left(\frac{2\pi}{L}v\right) - \cos\left(\frac{2\pi}{L}u\right)\right]dv$$

This is messy but easily doable, giving

$$\begin{aligned} P_{ex}(v)\,dv &= \frac{1}{2L}\left[\cos\left(\frac{\pi}{L}v\right) + \cos\left(\frac{3\pi}{L}v\right) - \frac{3}{2\pi}\sin\left(\frac{\pi}{L}v\right) + \right. \\ &\quad \left.\frac{7}{6\pi}\sin\left(\frac{3\pi}{L}v\right)\right] - \frac{v}{L^2}\cos\left(\frac{\pi}{L}v\right)\cos\left(\frac{2\pi}{L}v\right) \end{aligned}$$

Including for $-L \le v < 0$,

$$\begin{aligned} P_{ex}(v)\,dv &= \frac{1}{2L}\left[\cos\left(\frac{\pi}{L}v\right) + \cos\left(\frac{3\pi}{L}v\right) - \frac{3}{2\pi}\sin\left(\frac{\pi}{L}|v|\right) + \right. \\ &\quad \left.\frac{7}{6\pi}\sin\left(\frac{3\pi}{L}|v|\right)\right] - \frac{|v|}{L^2}\cos\left(\frac{\pi}{L}v\right)\cos\left(\frac{2\pi}{L}v\right) \end{aligned}$$

The symmetric probability is

$$P_S(v)\,dv = P(v)\,dv + P_{ex}(v)\,dv$$

(c) For antisymmetric spatial states, everything is the same as the symmetric case, except that the exchange contribution comes in with a $-$ sign.

$$P_A(v)\,dv = P(v)\,dv - P_{ex}(v)\,dv$$

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(a) For the ground state, there is no spatial antisymmetric state. Therefore the only possibility is a symmetric spatial part and an antisymmetric (singlet) spin part:

$$|\varphi_{00}^{SA}\rangle = |00\rangle \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)$$

The energy eigenvalue is $E_{00} = \hbar\omega$. The first excited state has four possibilities. First, with the singlet again

$$|\varphi_{10}^{SA}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right)\frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)$$

Then the triplet spin combinations:

$$|\varphi_{10}^{AS}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)|{+}{+}\rangle$$

$$|\varphi_{10}^{AS}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\frac{1}{\sqrt{2}}\left(|{+}{-}\rangle + |{-}{+}\rangle\right)$$

$$|\varphi_{10}^{AS}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)|{-}{-}\rangle$$

This is 4-fold degenerate with $E_{10} = 2\hbar\omega$.

(b) The antisymmetric spatial states have a large $\langle(x_1 - x_2)^2\rangle$ compared to the symmetric spatial states. In every state, $\langle(x_1 - x_2)^2\rangle > 0$. Therefore, since $H' = \frac{1}{2}\alpha(x_1 - x_2)^2$ and $\alpha > 0$, and since $E^{(1)} = \langle H'\rangle$, we have

- The ground state energy will be raised by a small amount.
- The excited state energy for the symmetric spatial combination, the singlet, will also be raised by a small amount.

- The triplet states will have their energy raised by a small amount, but this amount will be larger than the amount by which the singlet is raised.

The 4-fold degenerate state will be split into a singlet state plus a 3-fold degenerate triplet.