1. (50 points) In a H atom, you have a proton and an electron, both which are spin-$\frac{1}{2}$ particles. You describe the spin states by listing the proton spin first, the electron second:

$$|++\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |+-\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-+\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |--\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

You also have the usual spin operators $\vec{S}_p$ and $\vec{S}_e$, where the spin component operators affect only the spin of the appropriate particle. For example,

$$S_{pz}|+-\rangle = \tfrac{\hbar}{2}|+-\rangle, \quad S_{ez}|+-\rangle = -\tfrac{\hbar}{2}|+-\rangle$$

$$S_{py}|+-\rangle = i\tfrac{\hbar}{2}|--\rangle, \quad S_{ey}|+-\rangle = -i\tfrac{\hbar}{2}|++\rangle$$

and so forth.

(a) Construct the 4×4 matrix representations of S_{px}, S_{py}, S_{pz}, S_{ex}, S_{ey}, and S_{ez}, in the basis of the vectors listed above.

Answer: This is just a matter of calculating the matrix elements $\langle ++|S_{px}|++\rangle$ and so forth. Doing this,

$$S_{px} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad S_{ex} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_{py} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad S_{ey} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$S_{pz} \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_{ez} \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(b) Now look at the *total spin* operator $\vec{J} = \vec{S}_p + \vec{S}_e$. Construct the 4×4 matrix representations of J_z and J^2. Then find the eigenvectors common to J_z and J^2, and the corresponding eigenvalues of J_z and J^2.

Answer:

$$J_x = S_{px} + S_{ex} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$J_y = S_{py} + S_{ey} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}$$

$$J_z = S_{pz} + S_{ez} \doteq \frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$J^2 = J_x^2 + J_y^2 + J_z^2 \doteq \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

J_z is already diagonal, with eigenvalues $\hbar$, 0, $-\hbar$; with 0 doubly degenerate. And it's exactly in that degenerate subspace where J^2 is not diagonal. So we need to diagonalize that subspace to get the common eigenvectors:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \qquad \Rightarrow \qquad \lambda = 0, 2$$

$\lambda = 2$ means an eigenvalue of $2\hbar^2$ for J^2 and corresponds to the eigenvector

$$\frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = 0$ means an eigenvalue of 0 for J^2 and corresponds to the eigenvector

$$\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(c) Find the commutators $[J_x, J_y]$ and cyclic permutations, and $[J_z, J^2]$. Don't use matrices, but only the known commutation relationships involving the $\vec{S}_p$ and $\vec{S}_e$ operators. Do your commutators agree with what you would expect for angular momentum? What does this means about the common eigenstates of J_z and J^2? Interpret your results from part (b) accordingly—what j and m_j values do each eigenvector correspond to?

Answer: All operators associated with p commute with all operators associated with e, since they have nothing to do with each other. Therefore

$$\begin{aligned} [J_x, J_y] &= [S_{px} + S_{ex}, S_{py} + S_{ey}] = [S_{px}, S_{py}] + [S_{ex}, S_{ey}] \\ &= i\hbar S_{pz} + i\hbar S_{ez} = i\hbar J_z \end{aligned}$$

as would be expected for angular momentum. The cyclic permutations also give exactly what would be expected for angular momentum. Also,

$$\begin{aligned} [J_z, J^2] &= \left[S_{pz} + S_{ez}, (S_{px} + S_{ex})^2 + (S_{py} + S_{ey})^2 + (S_{pz} + S_{ez})^2\right] \\ &= \left[S_{pz}, S_{px}^2\right] + 2S_{ex}[S_{pz}, S_{px}] + \left[S_{pz}, S_{py}^2\right] + 2S_{ey}[S_{pz}, S_{py}] + \\ &\quad \left[S_{ez}, S_{ex}^2\right] + 2S_{px}[S_{ez}, S_{ex}] + \left[S_{ez}, S_{ey}^2\right] + 2S_{py}[S_{ez}, S_{ey}] \\ &= 0 \end{aligned}$$

These are exactly the commutation relationships angular momentum operators should have. Therefore they should have the spectrum of angular momentum operators, with eigenstates $|jm_j\rangle$ such that $J^2|jm_j\rangle = j(j+1)\hbar^2|jm_j\rangle$ and $J_z|jm_j\rangle = m_j\hbar|jm_j\rangle$.

Looking at the answers to (b), we can see three eigenstates with total angular momentum $2\hbar^2$. These must be $j = 1$ states with $m_j = 1, 0, -1$:

$$|11\rangle = |++\rangle, \qquad |10\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right), \qquad |1\,{-1}\rangle = |--\rangle$$

The remaining eigenstate has a total angular momentum of 0, so it is the state with $j = 0$ and $m_j = 0$:

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$