1. (50 points) A black hole's event horizon defines a surface in space; no information about what is within that surface can be transmitted to the outside world. Since entropy is a measure of missing information, it won't surprise you that the entropy of a black hole is proportional to its area. In fact, with the Planck length $L_p = \sqrt{G\hbar/c^3}$,

$$\frac{S}{k} = \frac{A}{4L_p^2} = \frac{c^3}{4G\hbar}A$$

where A is the surface area. In the following, recall that a non-rotating Schwarzschild black hole has radius $R = 2GM/c^2$, where M is the mass of the black hole. Also, in your results, the only physical constants that appear should be k, G, $\hbar$, and c, and please simplify your results as much as possible.

Note: This is a classic question, which means you can find almost all the solutions to the following online. Nonetheless, I expect you to do this yourself, without consulting online sources. *Consult me if you have any questions;* I will make sure you're on the right track.

(a) The energy of a black hole is $U = Mc^2$. Calculate the temperature of a Schwarzschild black hole with mass M.

Answer: Putting things together,

$$S = \frac{4\pi kG}{\hbar c}M^2 = \frac{4\pi kG}{\hbar c^5}U^2$$

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{8\pi kG}{\hbar c^5}U = \frac{8\pi kG}{\hbar c^3}M \qquad \Rightarrow \qquad T = \frac{\hbar c^3}{8\pi kG}\frac{1}{M}$$

Note that the smaller the black hole, the higher the temperature.

(b) A black hole is a perfect blackbody, which will radiate energy due to its temperature. Calculate the rate of evaporation of a Schwarzschild black hole of mass M. Assume that only photons will be radiated. Note that the Stefan-Boltzmann constant $\sigma = \pi^2 k^4/60\hbar^3 c^2$.

Answer: The power radiated by a blackbody is $\sigma e A T^4$. For a perfect blackbody $e = 1$. Since there's no compression-expansion work in outer space, $dU = đQ$. Therefore

$$\frac{dU}{dt} = c^2\frac{dM}{dt} = -\sigma A T^4$$

The negative sign is because the energy/mass of the black hole will *decrease* with the evaporation. Putting in everything,

$$\frac{dM}{dt} = -\frac{\hbar c^4}{2^{10} 15\pi G^2} \frac{1}{M^2} = -r\,\frac{1}{M^2}$$

where I've called the constant up front r.

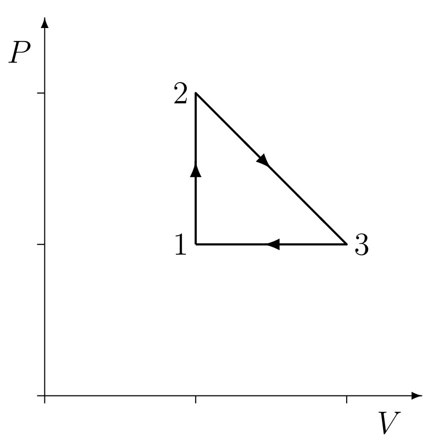
(c) Calculate $t_{ev}(M)$, the time it takes for a Schwarzschild black hole to totally evaporate. Calculate this value for a typical stellar remnant black hole with $M = 20M_\odot$. ($M_\odot$ is a solar mass.)

Answer: Rewrite the rate as $M^2 dM = -r\,dt$. We can integrate both sides, starting from $t = 0$ where the mass is M and going until t_{ev} where the mass is 0:

$$\int_M^0 dM\, M^2 = -r \int_0^{t_{ev}} \qquad \Rightarrow \qquad t_{ev} = \frac{M^3}{3r} = \frac{2^{10} 5\pi G^2}{\hbar c^4} M^3$$

Putting in the numbers for $M = 20M_\odot$, we get $t_{ev} = 5.3 \times 10^{78}$ s, or 1.7×10^{71} years. A *lot* longer than the age of the universe!

2. (50 points) You have a heat engine that uses a monatomic ideal gas as a working fluid, and that goes through the following cycle, where $P_2 = 2P_1$ and $V_3 = 2V_1$.



(a) Calculate the efficiency of the heat engine. Note, however, that you should not assume that the heat transfer to the engine during $2 \to 3$ is always the same sign. Find out if there are regions on the line segment where $Q < 0$, and if so, account for it as part of Q_h or Q_c accordingly.

Answer: For $1 \to 2$,

$$W_{12} = 0 \quad Q_{12} = \Delta U - W = C_v \Delta T = \frac{3}{2}\left(P_2 V_1 - P_1 V_1\right) = \frac{3}{2} P_1 V_1 > 0$$

For $3 \to 1$,

$$\begin{aligned} W_{31} &= -P\Delta V = P_1 V_1 \\ Q_{31} &= \Delta U - W = C_p \Delta T = \frac{5}{2}\left(P_1 V_1 - P_1 V_3\right) = -\frac{5}{2} P_1 V_1 < 0 \end{aligned}$$

For $2 \to 3$, the equation of the path is

$$P(V) = P_1 \left(3 - \frac{V}{V_1}\right)$$

Therefore, since $PV = NkT$, on this path,

$$P_1 \left(3 - \frac{V}{V_1}\right) V = NkT \quad \Rightarrow \quad P_1 \left(3 - \frac{2V}{V_1}\right) dV = Nk\, dT$$

Now, since $đW = -pdV$ and $đQ = dU - đW$,

$$\begin{aligned} đQ &= \frac{3}{2} Nk\, dT + PdV = P_1 \left[\frac{3}{2}\left(3 - \frac{2V}{V_1}\right) + \left(3 - \frac{V}{V_1}\right)\right] dV \\ &= \frac{P_1}{2}\left[15 - 8\frac{V}{V_1}\right] dV \end{aligned}$$

For V increasing from V_1 until $2V_1$, $dV > 0$, and this means that $đQ > 0$ for $V_1 < V < \frac{15}{8}V_1$, and $đQ < 0$ for $\frac{15}{8}V_1 < V < 2V_1$. The two line segments must be accounted for separately, as part of Q_h and as part of Q_c. The positive and negative parts are:

$$Q_+ = \frac{P_1}{2}\int_{V_1}^{\frac{15}{8}V_1} dV \left(15 - 8\frac{V}{V_1}\right) = \frac{49}{32} P_1 V_1 = 1.53\, P_1 V_1$$

$$Q_- = \frac{P_1}{2}\int_{\frac{15}{8}V_1}^{2V_1} dV \left(15 - 8\frac{V}{V_1}\right) = -\frac{1}{32} P_1 V_1 = -0.03\, P_1 V_1$$

The work done from $2 \to 3$ is straightforward; $W_{23} = -\frac{3}{2}P_1V_1$.

Adding positive and negative heats up separately,

$$Q_h = Q_{12} + Q_+ = \frac{97}{32}P_1V_1 = 3.03\,P_1V_1$$

$$Q_c = -Q_{12} - Q_- = \frac{81}{32}P_1V_1 = 2.53\,P_1V_1$$

$$W = -W_{12} - W_{23} - W_{31} = \frac{1}{2}P_1V_1 = 0.50\,P_1V_1$$

The efficiency is

$$e = \frac{W}{Q_h} = \frac{16}{97} = 0.165$$

(b) Find the efficiency of a Carnot engine operating between the maximum and minimum temperatures on this cycle.

Answer: We can start by remembering that isotherms are hyperbolas, and that the hyperbolas move away from the axes as T increases. Therefore, from symmetry, the maximum temperature on the cycle is midpoint on the $2 \to 3$ part. This is at $P = \frac{3}{2}P_1$ and $V = \frac{3}{2}V_1$, so that

$$T_h = \frac{9P_1V_1}{4Nk}$$

The minimum temperature is at point 1, so $T_c = P_1V_1/Nk$. The maximum efficiency is therefore

$$e_{\text{max}} = 1 - \frac{T_c}{T_h} = \frac{5}{9} = 0.556$$